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IMPROVING THE PERFORMANCE OF THIRD GENERATION WIRELESS COMMUNICATION SYSTEMS

REMCO VAN DER HOFSTAD * AND
MARTEN J. KLOK,* ** *Delft UT*

Email address: rhofstad@win.tue.nl, MKlok@ortec.nl

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Abstract

The third generation (3G) mobile communication system uses a technique called code division multiple access (CDMA), in which multiple users use the same frequency and time domain. The data signals of the users are distinguished using codes. When there are many users, interference deteriorates the quality of the system. For more efficient use of resources, one wishes to allow more users to transmit simultaneously, by using algorithms that utilize the structure of the CDMA system more effectively than the simple Matched Filter (MF) system used in the proposed 3G systems. In this paper, we investigate an advanced algorithm called hard-decision parallel interference cancellation (HD-PIC), in which estimates of the interfering signals are used to improve the quality of the signal of the desired user. We compare HD-PIC with MF in a simple case, where the only two parameters are the number of users and the length of the coding sequences. We focus on the exponential rate for the probability of a bit-error, explain the relevance of this parameter, and investigate how it scales when the number of users grows large. We also review extensions of our results proved elsewhere showing that in HD-PIC, more users can transmit without errors than in the MF system.

1. Introduction and results

1.1. Introduction

In this paper, we study the performance in the third generation (3G) mobile communication system, which is based on a technique called code division multiple access (CDMA). In the first generation (1G) of mobile communication systems, each users is assigned a frequency, like radio. By the increase of the number of cell phones, this system is no longer possible. In the second generation (2G), which is often called GSM, different users send and receive on the same frequency, and the signals are separated by assigning each user a time window in which he or she can transmit. This system is not very efficient, since it is hard to adapt to changing numbers of users. Therefore, the allowed time and frequency window is not used optimally. In the third generation systems, users are distinguished by their coding sequences, which gives a flexible system, in which the amount of bandwidth is used more efficiently.

We start by explaining the mathematical model behind CDMA. Suppose that k users transmit data across a channel simultaneously. In order to do so, each user multiplies his data signal by an individual coding sequence. At the receiver, the signal of the m^{th} ($1 \leq m \leq k$) user can be retrieved by taking the inner product of the transformed total signal and the m^{th} coding sequence. When the coding sequences are orthogonal, all data that does not originate from the m^{th} user will be annihilated. Avoiding interference at any cost is expensive and not efficient, and in practice, almost-orthogonal codes (like pseudo-random codes) are used. The technique of coding signals in order to transmit various signals simultaneously is known as *code division multiple access* (CDMA), see for example [26]. We next give a mathematical description of

* Postal address: Faculty ITS, Delft University of Technology, Mekelweg 4, 2628 CD Delft, the Netherlands.
* Current address: Department of Mathematics and Computer Science, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands
** Current address: ORTEC BV, Orlyplein 145c, 1043 DV Amsterdam, The Netherlands

CDMA systems. We define the data signal $b_m(t)$ of the $m^{\rm th}$ user as $b_m(t) = b_{m\,\lceil t/T\rceil}$, for $0 \le m \le k-1$, where

$$b_m = (\dots, b_{m,-1}, b_{m0}, b_{m1}, \dots) \in \{-1,+1\}^{\mathbb{Z}}$$

and where for $x \in \mathbb{R}$, $\lceil x\rceil$ denotes the smallest integer larger than or equal to x. For each m, $0 \le m \le k-1$, we have a sequence $a_m = (\dots, a_{m,-1}, a_{m0}, a_{m1}, \dots) \in \{-1,+1\}^{\mathbb{Z}}$ and we put $a_m(t) = a_{m\,\lceil t/T_c\rceil}$, where $T_c = T/n$, for some integer n. In practice, the value of n ranges from $32-512$. The transmitted coded signal of the $m^{\rm th}$ user is then

$$s_m(t) = \sqrt{2P_m}\, b_m(t) a_m(t) \cos(\omega_c t), \quad 0 \le m \le k-1, \tag{1}$$

where P_m is the power of the $m^{\rm th}$ user and ω_c the carrier frequency. The factor $\cos(\omega_c t)$ is used to transmit the signal at frequency ω_c. The variable n is often called the processing gain and this can be understood by explaining the military origins of CDMA, see [28]; CDMA turns out to reduce the signal-to-noise ratio of the transmitter and the jammer by a factor n. The code $a_m(t)$ is known to the transmitter (e.g., the mobile phone of the transmitting person) and to the base station. Transmission from phone to base station is called uplink[1], whereas transmission in the other direction is called downlink. We will continue with the mathematical description. The total transmitted signal is given by

$$r(t) = \sum_{j=0}^{k-1} s_j(t). \tag{2}$$

In practice, the signals do not need to be synchronized, i.e., it is not necessary that all users transmit using the same time grid. However, for technical reasons we do assume so. For simplicity, we assume that the noise on the channel is negligible, so that the transmitted and received signals agree.

To retrieve the data bit b_{m0}, the signal $r(t)$ is multiplied by $a_m(t)\cos\omega_c t$ and then averaged over $[0,T]$. In practice $\omega_c T_c$ is large; the carrier frequency (ω_c) is much higher than the bandwidth (2Δ) in electrotechnical terms. For simplicity, we pick $\omega_c T_c = \pi f_c$, where $f_c \in \mathbb{N}$ to get (cf. [10], Eqn. (3)-(4))

$$\frac{1}{T}\int_0^T r(t)a_m(t)\cos(\omega_c t)\,dt = \frac{1}{2}\sqrt{2P_m}b_{m1} + \sum_{\substack{j=0\\ j\neq m}}^{k-1} \frac{1}{2}\sqrt{2P_j}b_{j1}\frac{1}{n}\sum_{i=1}^{n} a_{ji}a_{mi}. \tag{3}$$

The above procedure is the one used in 3G systems, and is often referred to as *Matched Filter* (MF). As is seen from (3) the decoded signal consists of the desired bit and interference due to the other users. In the ideal situation the vectors $(a_{m1}, \dots, a_{m,n})$ and $(a_{j1}, \dots, a_{j,n})$, $j \neq m$, would be orthogonal, so that $\sum_{i=1}^n a_{ji}a_{mi} = 0$. In practice, the a-sequences are generated by a random number generator. To model the pseudo-random sequence a, let A_{mi}, $m = 0,1,\dots,k-1$, $i = 1,2,\dots,n$, be an array of independent and identically distributed (i.i.d.) random variables with distribution

$$\mathrm{P}(A_{mi} = +1) = \mathrm{P}(A_{mi} = -1) = 1/2. \tag{4}$$

In practice, the a-sequences are not chosen as i.i.d. sequences. Rather, they are carefully chosen to have good correlation properties. Examples are Gold sequences [8] or Kasami sequences [17]. Sometimes better performance can be achieved for well-chosen deterministic codes. However, it is common in the literature to use random sequences, so that a detailed analysis of the performance in the system is possible. Assuming the coding sequences to be random, we model the signal of (3) as

$$\frac{1}{2}\sqrt{2P_m}b_{m1} + \sum_{\substack{j=0\\ j\neq m}}^{k-1} \frac{1}{2}\sqrt{2P_j}b_{j1}\frac{1}{n}\sum_{i=1}^{n} A_{ji}A_{mi}.$$

[1]The term uplink is a carry-over from satellite communications

An estimator for b_{m1} is given by

$$\hat{b}_{m1}^{(1)} = \text{sgnr}\Big\{\frac{1}{2}\sqrt{2P_m}b_{m1} + \sum_{\substack{j=0\\ j\neq m}}^{k-1}\frac{1}{2}\sqrt{2P_j}b_{j1}\frac{1}{n}\sum_{i=1}^{n}A_{ji}A_{mi}\Big\},$$

where, for $x \in \mathbb{R}$, the randomized sign-function is defined as

$$\text{sgnr}(x) = \begin{cases} +1, & x > 0, \\ U, & x = 0, \\ -1, & x < 0, \end{cases} \quad \text{with } \mathrm{P}(U = -1) = \mathrm{P}(U = +1) = 1/2. \tag{5}$$

In the presence of noise, the randomized sign-function never needs to be used. However, since, for simplicity, we assume the channel to be noise-free, we need to introduce it. The random variable U is independent of all other random variables in the system and every time we need the sgnr-function another independent trial U is performed.

The above system is called the Matched Filter (MF) system, and will be used in the 3G telecommunication systems. The superscript $^{(1)}$ indicates that $\hat{b}_{m1}^{(1)}$ is a tentative decision. Below we will show how we can improve the estimates. We are interested in the probability of a bit-error, i.e., in $\mathrm{P}(\hat{b}_{m1}^{(1)} \neq b_{m1})$, since this is a good measure for the quality of the system. We can investigate this probability using Gaussian approximations [14], [18], [23], or other approximation techniques [22], [30], [33]. Since the receiver described above is designed for unspecified random noise, a signal-to-interference threshold limits performance. However, interference experienced in a CDMA system is different from completely random noise, and this fact can be exploited to achieve better performance. Receivers that exploit information of the system (mainly the cross correlations of the codes) are often denoted by *advanced* receivers or *multiuser detection* receivers. For an overview, see [24] and [3].

The best known multiuser detection systems technique is a *maximum likelihood* estimator, introduced in [35], which obtains jointly optimum decisions for all users using maximum likelihood detection. Unfortunately, this technique is of such high complexity that it cannot be performed real-time. A more straightforward technique is called *interference cancellation* (IC). The idea is that we try to cancel the interference due to the other users (i.e., the users with subscript $j \neq m$). According to [27], IC is seen as "the most promising and the most practical technique for base-station (uplink)receivers". For mobile interfaces (downlink), such as mobile phones, IC is not practical because it demands that each mobile interface has access to all codes. This is clearly not desirable for security reasons. However, orthogonal codes can be used downlink, since unlike for uplink communication, downlink the signals will be synchronized, so that reducing interference is of less concern. Also, blind estimation schemes exist that can improve performance in the downlink communication, see [32]. These blind schemes do not require a priori information on the structure of the interfering signals.

IC comes in many flavours. We will now explain the multiuser detection system that we will focus on in this paper, which is called *hard decision parallel interference cancellation* (HD-PIC) (see [2], [4], [16], [21], [34] and the references therein). HD-PIC is seen as "the most promising IC scheme for the uplink" (cf. [26]). A variation is soft decision PIC, which we have investigated in [10], and will be explained below. Also serial IC is proposed (cf. [15], [25], [29]), in which the interference is cancelled one user at a time, while in the parallel approach all users attempt to cancel their interference at the same time. We will not treat the serial IC system in this paper.

The hard decision procedure is described below. Various techniques exist that are able to estimate the powers P_m with high accuracy (cf. [27], Sect. 7.3), so we may assume that the powers P_m are known. When it turns out that the power cannot be estimated reliably, one can instead estimate $\sqrt{P_m}b_{m1}$, resulting in the soft-decision procedure. In this paper we assume that we can estimate the power superiorly compared to the bits. We estimate the data signal $s_j(t)$ for $t \in [0, T]$ by (recall (1))

$$\hat{s}_j^{(1)}(t) \quad = \quad \sqrt{2P_j}\,\hat{b}_{j1}^{(1)}\,a_j(t)\cos(\omega_c t). \tag{6}$$

Then we estimate the total interference for the m^{th} user in $r(t)$ due to the other users by (recall

(2))

$$\hat{r}_m^{(1)}(t) = \sum_{\substack{j=0\\ j\neq m}}^{k-1} \hat{s}_j^{(1)}(t)$$

We use the above to find a better estimate of the data bit b_{m1}:

$$\begin{aligned}
\hat{b}_{m1}^{(2)} &= \text{sgnr}\left\{\frac{1}{T}\int_0^T (r(t) - \hat{r}_m^{(1)}(t))a_m(t)\cos(\omega_c t)dt\right\} \\
&= \text{sgnr}\left\{\frac{1}{2}\sqrt{2P_m}b_{m1} + \sum_{\substack{j=0\\ j\neq m}}^{k-1}\frac{1}{2}\sqrt{2P_j}\left(\frac{1}{n}\sum_{i=1}^n A_{ji}A_{mi}\right)\left(b_{j1} - \hat{b}_{j1}^{(1)}\right)\right\}. \qquad (7)
\end{aligned}$$

We are now interested in $\mathrm{P}(\hat{b}_{m1}^{(2)} \neq b_{m1})$, which is the probability of a bit-error after one stage of interference cancellation. We will see that this probability is indeed smaller than $\mathrm{P}(\hat{b}_{m1}^{(1)} \neq b_{m1})$, the probability of a bit-error without cancellation. This motivates a repetition of the previous procedure. We obtain, similarly to (7), the estimates $\hat{b}_{m1}^{(s)}$:

$$\hat{b}_{m1}^{(s)} = \text{sgnr}\left\{\frac{1}{2}\sqrt{2P_m}b_{m1} + \sum_{\substack{j=0\\ j\neq m}}^{k-1}\frac{1}{2}\sqrt{2P_j}\left(\frac{1}{n}\sum_{i=1}^n A_{ji}A_{mi}\right)\left(b_{j1} - \hat{b}_{j1}^{(s-1)}\right)\right\}.$$

This is called *multistage HD-PIC*. When we have applied s steps of interference cancellation we speak of s-stage HD-PIC and the corresponding bit-error probability is $\mathrm{P}(\hat{b}_{m1}^{(s+1)} \neq b_{m1})$.

In this paper, we will investigate the bit-error probability, and see to what extent HD-PIC decreases the bit-error probability. For completeness, let us also describe soft decision parallel interference cancellation (SD-PIC). In this case, rather than estimating $\sqrt{2P_j}\, b_{j1}$ by $\sqrt{2P_j}\, \hat{b}_{j1}^{(1)}$ in (6), we estimate $\sqrt{2P_j}b_{j1}$ in (6) by

$$\sqrt{2P_j}b_{j1} + \sum_{\substack{l=0\\ l\neq j}}^{k-1}\sqrt{2P_l}b_{l1}\frac{1}{n}\sum_{i=1}^n A_{li}A_{ji}, \qquad (8)$$

and then the computation in (7) is performed with this new estimate. This is 2-stage SD-PIC, and for multistage SD-PIC the above steps are repeated. SD-PIC has the main advantage that the powers do not need to be known. Moreover, the idea behind SD-PIC is that when $Z_j^{(1)}$ is close to 0, then we are not very sure that we have estimated the bit b_{j1} correctly, and therefore, the estimator should have less weight in the IC scheme.

1.2. Reformulation of the problem

We can write the probability of a bit-error in a more convenient way, and show that without loss of generality, we may assume that $b_{j1} = 1$ for all $j = 0, \ldots, k-1$. Namely, because $b_{mi}^2 = 1$, we have

$$\frac{1}{2}\sqrt{2P_m}b_{m1} + \sum_{\substack{j=0\\ j\neq m}}^{k-1}\frac{1}{2}\sqrt{2P_j}b_{j1}\frac{1}{n}\sum_{i=1}^n A_{ji}A_{mi} = b_{m1}\left(\frac{1}{2}\sqrt{2P_m} + \sum_{\substack{j=0\\ j\neq m}}^{k-1}\frac{1}{2}\sqrt{2P_j}\frac{1}{n}\sum_{i=1}^n b_{j1}A_{ji}b_{m1}A_{mi}\right).$$

Since $A_{ji} \stackrel{d}{=} b_{j1}A_{ji}$, we have

$$\mathrm{P}(\hat{b}_{m1}^{(1)} \neq b_{m1}) = \mathrm{P}\left(\frac{\hat{b}_{m1}^{(1)}}{b_{m1}} \neq 1\right) = \mathrm{P}(\text{sgnr}(Z_m^{(1)}) < 0) = \mathrm{P}(Z_m^{(1)} < 0) + \frac{1}{2}\mathrm{P}(Z_m^{(1)} = 0),$$

where $Z_m^{(1)}$, for $0 \leq m \leq k-1$, is defined as

$$Z_m^{(1)} = \frac{1}{2}\sqrt{2P_m} + \sum_{\substack{j=0\\ j\neq m}}^{k-1}\frac{1}{2}\sqrt{2P_j}\frac{1}{n}\sum_{i=1}^n A_{ji}A_{mi}.$$

Therefore, the bit-error probability is independent of the actual values of b_{j1}, and we may assume that $b_{j1} = 1$ for all $j = 0, \dots, k-1$. In a similar way, we define for $s \geq 2$ and $0 \leq m \leq k-1$,

$$Z_m^{(s)} = \frac{1}{2}\sqrt{2P_m} + \sum_{\substack{j=0 \\ j\neq m}}^{k-1} \left(\frac{1}{n} \sum_{i=1}^{n} \frac{1}{2}\sqrt{2P_j} A_{ji} A_{mi} \right) [1 - \mathrm{sgnr}(Z_j^{(s-1)})], \tag{9}$$

to obtain

$$\mathrm{P}(\hat{b}_{m1}^{(s)} \neq b_{m1}) = \mathrm{P}(Z_m^{(s)} < 0) + \frac{1}{2}\mathrm{P}(Z_m^{(s)} = 0). \tag{10}$$

This formula allows us to compute the exponential rate of the bit-error probability, as we will explain in more detail in the following section.

1.3. Results

In this paper, we describe the behaviour of the probability $\mathrm{P}(\hat{b}_{m1}^{(s)} \neq b_{m1})$ in the asymptotic case $n \to \infty$. These bit-errors are rare events as $n \to \infty$, therefore we focus on the rate at which this probability tends to zero, i.e., we investigate

$$H_k^{(s)} = -\lim_{n\to\infty} \frac{1}{n} \log \mathrm{P}(\hat{b}_{m1}^{(s)} \neq b_{m1}). \tag{11}$$

Clearly, the above exponential rate is a function of all the powers. To reduce the number of parameters in the system, and since we will be especially interested in the case where the number of users k is large, we will for simplicity assume that all powers are equal, and without loss of generality, we may then assume that all powers are equal to 2. In practice, this is called *perfect power control*. In this case, the only parameters in the system are n and k, and we will be interested in the limit when first n grows large, followed by k growing large. When the powers are equal, the random variables $Z_0^{(s)}, Z_1^{(s)}, \dots, Z_{k-1}^{(s)}$ are exchangeable, so that it suffices to consider the case where $m = 0$.

We will now explain the relevance of the exponential rate as a performance measure. Clearly, for large n, the bit-error probability can be written as $e^{-nH_k^{(s)}(1+o(1))}$. The bit-error probability is mainly characterised by its exponential rate, and a small increase of the exponential rate for n large leads to a large decrease of the bit-error probability. Therefore, we can think of the exponential rate of the bit-error probability as a convenient measure of performance, and one can compare the performance in two systems by comparing the exponential rates of their respective bit-error probabilities. From a practical point of view, this is indeed convenient, as the exponential rate is a single number, rather than a sequence of numbers such as the bit-error probabilities for various values of n. Therefore, for two systems with the same number of users, to compare the efficiency of the two systems, we can simply compare the two exponential rates of the bit-error probabilities to gain theoretical insight in the performance of the two systems. Moreover, even though the exponential rate only arises in the limit as $n \to \infty$, the asymptotic approximation that $\log \mathrm{P}(\hat{b}_{m1}^{(s)} \neq b_{m1}) = -nH_k^{(s)} + o(n)$ can also be used when k and n grow large simultaneously. In fact, the main results in this paper compute the asymptotics of $H_k^{(s)}$ for k large, and this asymptotics can be used directly to give bounds on the bit-error probability. See Section 1.5 for a more detailed explanation. In particular, we can show that there are values of $k = k_n$ such that the MF system, used in the current 3G systems, will have bit-errors with large probability, while the HD-PIC system for $s \geq 2$ does not. Finally, in [11, Section II.E], it is argued that the exponential rate of a bit-error, especially in *lightly loaded* systems (i.e., systems where the number of users is small compared to the code length n), is a better measure of performance that the more commonly used signal-to-noise ratio, which compares the mean and variance of the signal. Indeed, a signal-to-noise computation is often based on underlying Gaussian assumptions, which are not satisfied in the lightly loaded systems.

We now describe results concerning the exponential rates $H_k^{(s)}$. First of all, note that the probability that there exists a bit-error is bounded by

$$\mathrm{P}(\hat{b}_{m1}^{(s)} \neq b_{m1}) \leq \mathrm{P}(\exists 0 \leq j \leq k-1 : \hat{b}_{j1}^{(s)} \neq b_{j1}) \leq k\,\mathrm{P}(\hat{b}_{m1}^{(s)} \neq b_{m1}), \tag{12}$$

so that the exponential rate of the existence of at least one bit-error in stage s is also given by $H_k^{(s)}$ in (11). This implies that the map $s \mapsto H_k^{(s)}$ is non-decreasing. Indeed, to have a bit-error at stage s, it is necessary to have a bit-error at stage $s-1$. As a consequence of our results, we will see that $H_k^{(s)} > H_k^{(s-1)}$ when k is sufficiently large, so that multistage HD-PIC significantly decreases the probability of a bit-error.

For $k=1$, there is no interference due to other users and therefore in this case $\mathrm{P}(Z_0^{(s)} \leq 0) = 0$, for all s. For $k=2$, i.e., for *two* users something peculiar happens. It is readily seen that for $k=2$,

$$Z_0^{(1)} = 0 \qquad \text{if and only if} \qquad \frac{1}{n}\sum_{i=1}^{n} A_{1i}A_{2i} = -1$$

and $Z_0^{(1)} > 0$ otherwise. As a consequence, the event $Z_0^{(s)} = 0$ involves only one atom and we see that $\mathrm{P}(\mathrm{sgnr}(Z_0^{(s)}) < 0) = (1/2)^{n+1}$ and thus $H_2^{(s)} = \log 2$. In the remainder of the paper we will only consider the case that $k \geq 3$. In [10] it is shown that the exponential rate without interference cancellation $H_k^{(1)}$ (denoted there by I_k), for $k \geq 3$, is given by[1]

$$H_k^{(1)} = \frac{k-2}{2}\log\Big(\frac{k-2}{k-1}\Big) + \frac{k}{2}\log\Big(\frac{k}{k-1}\Big).$$

For $1/k \to 0$, a Taylor series expansion yields

$$H_k^{(1)} = \frac{1}{2k} + \mathcal{O}\Big(\frac{1}{k^2}\Big). \tag{13}$$

We can summarise this result by writing that

$$\mathrm{P}(\hat{b}_{m1}^{(s)} \neq b_{m1}) = e^{-\frac{n}{2k}(1+o(1))}, \tag{14}$$

where $o(1)$ converges to zero when first $n \to \infty$ and then $k \to \infty$. This result can also be seen as a kind of central limit theorem (CLT) result, since for k and n large the CLT states that with $\mathrm{E}\, Z_1^{(1)} = 1$, $\mathrm{var}(Z_1^{(1)}) = k/n$,

$$\log \mathrm{P}(\hat{b}_{m1}^{(1)} \neq b_{m1}) = (1+o(1)) \log Q(\sqrt{n/k}),$$

where $Q(x)$ is the probability that a standard Gaussian random variable exceeds the value x. Furthermore, it is well-known that on an exponential scale, we can approximate

$$Q(x) = e^{-x^2/2(1+o(1))},$$

so that we again arrive at (14).

For k large, $H_k^{(1)}$ converges to 0 as $1/(2k)$, which limits the number of users in the system. We will describe this in more detail in Section 1.5 below. We now describe our main results. We will start by computing the exponential rate when $s=2$ in the next theorem:

Theorem 1. *For every $k \geq 3$,*

$$H_k^{(2)} = -\lim_{n\to\infty} \frac{1}{n} \log \mathrm{P}(Z_0^{(2)} \leq 0)$$

exists, and is equal to

$$H_k^{(2)} = \min_{1 \leq r \leq k-1} H_{k,r}^{(2)}, \qquad \textit{where} \qquad H_{k,r}^{(2)} = \sup_{\underline{t} \in (-\infty,0]^2} \{-\log h_{k,r}(\underline{t})\}, \tag{15}$$

with

$$h_{k,r}(\underline{t}) = 2^{-r} \sum_{j=-r,-r+2}^{r} \binom{r}{\frac{r+j}{2}} e^{t_1(j+j^2)+t_2(1+2j)} (\cosh t_1 j)^{k-r-1}. \tag{16}$$

[1] A different definition of the sign function is used there, but this does not influence the results.

The above result shows that for k fixed, the bit-error probability is exponentially small, and gives an explicit expression for its exponential rate. The variable r appearing in (15) arises as the number of bit-errors in the first stage MF system. In Section 1.4, we give numerical results for $H_k^{(2)}$ showing that $H_k^{(2)} > H_k^{(1)}$ for $k \geq 3$, so that HD-PIC substantially improves performance compared to the MF system used in the current 3G wireless communication systems.

In Theorem 1, the description of $H_k^{(2)}$ is explicit, but already rather involved. In principle, a similar result should hold for all $s \geq 2$, but the formula would not be very illuminating, as it wold involve minima over various variables related to r and suprema over variables related to t, s. Therefore, instead of computing the explicit value of $H_k^{(s)}$, we compute its asymptotic when k becomes large. In fact, from a practical point of view, one would like to allow for as many users as possible in the system, so that one is naturally lead to the problem of asymptotics for k large. In Section 1.5, we will describe consequences of this asymptotic behaviour on the number of users allowed so that no bit-error arises. We are now ready to characterize the asymptotic behaviour of $H_k^{(s)}$. Our main result is the following theorem:

Theorem 2. *Fix $1 \leq s < \infty$. As $k \to \infty$,*

$$H_k^{(s)} = \frac{s\sqrt[s]{4}}{8\sqrt[s]{k}}\left(1 + \mathcal{O}\left(\frac{1}{\sqrt[s]{k}}\right)\right). \tag{17}$$

Note that for $k = 1$, we obtain (13), while for $s = 2$, we obtain

$$H_k^{(2)} = \frac{1}{2\sqrt{k}} + O(\frac{1}{k}). \tag{18}$$

When we think of $H_k^{(s)}$ as a performance measure of the system, we see that for very large k, each application of HD-PIC dramatically increases the exponential rate of the bit-error probability, thus dramatically decreasing the bit-error probability. We will describe a number of consequences of the above result in Section 1.5 below.

1.4. Heuristics and numerical work

In this section, we give a heuristic explanation of the main results in Theorems 1–2. As shown in Theorem 1, for $s = 2$, we are able to compute the exponential rate in two steps. First we calculate the exponential rate of the probability of the event that $\{Z_0^{(2)} \leq 0\}$, intersected with the events $\{\mathrm{sgnr}(Z_m^{(1)}) < 0\}$ for $m = 1, \ldots, r$ and the events $\{\mathrm{sgnr}(Z_m^{(1)}) > 0\}$ for $m = r+1, \ldots, k-1$, i.e.,

$$H_{k,r}^{(2)} = -\lim_{n\to\infty} \frac{1}{n} \log \mathrm{P}\left(\max_{1\leq m\leq r} \mathrm{sgnr}(Z_m^{(1)}) < 0,\ \min_{m\geq r+1} \mathrm{sgnr}(Z_m^{(1)}) > 0,\ \mathrm{sgnr}(Z_0^{(2)}) < 0\right). \tag{19}$$

In (19), the variable r stands for the number of bit-errors in the MF stage $s = 1$. We obtain from (9) on the intersection with the events $\{\max_{1\leq m\leq r} \mathrm{sgnr}(Z_m^{(1)}) < 0\}$ and $\{\min_{r+1\leq m\leq k-1} \mathrm{sgnr}(Z_m^{(1)}) > 0\}$ that $Z_0^{(2)}$ has the form

$$\bar{Z}_0^{(2)} = 1 + 2\sum_{j=1}^{r} \frac{1}{n} \sum_{i=1}^{n} A_{ji}A_{1i},$$

where the bar denotes that $\mathrm{sgnr}(Z_m^{(1)})$ is replaced by its corresponding value. We will show that the rate in (19) equals

$$-\lim_{n\to\infty} \frac{1}{n} \log \mathrm{P}\left(\max_{1\leq m\leq r} Z_m^{(1)} \leq 0,\ \min_{m\geq r+1} Z_m^{(1)} \geq 0,\ \mathrm{sgnr}(\bar{Z}_0^{(2)}) < 0\right). \tag{20}$$

We specify the latter rate as the solution of a 2-dimensional minimization problem given in the second equality in (15). The desired rate $H_k^{(2)}$ is therefore given by the first equality in (15).

k	r_k
{3,...,9}	1
{10,...,26}	2
{27,...,51}	3
{52,...,84}	4
{85,...,125}	5
{126,...,174}	6
{175,...,231}	7

TABLE 1: Optimal r

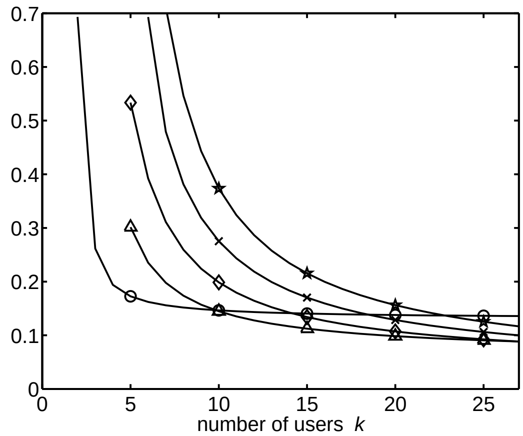

FIGURE 2: Exponential rates $H^{(2)}_{k,r}$ for $r = 1, \dots, 5$ indicated with $\circ, \triangle, \diamond, \times, \star$ respectively.

The above result specifies $H^{(2)}_k$ for fixed k. We now heuristically explain Theorem 2. We prove that for $r \to \infty$ and $r/k \to 0$,

$$H^{(2)}_{k,r} \approx \frac{r}{2k} + \frac{1}{8r}, \tag{21}$$

from where we prove (17) for $s = 2$.

We will next give a heuristic explanation of (17). For simplicity we stick to $s = 2$ and argue that (21) holds. The case $s > 2$ can be argued similarly. Observe that $\mathrm{E}\, Z^{(1)}_m = 1$, so that in the large deviation setting one expects $Z^{(1)}_m$ to be positive. Indeed, it turns out that the event $\{\min_{m \geq r+1} Z^{(1)}_m \geq 0\}$ in (20) does not contribute to the rate. Independence of $\{Z^{(1)}_1, \dots, Z^{(1)}_r, \bar{Z}^{(2)}_0\}$ is clearly false for finite k, but we can prove asymptotic independence as k tends to ∞. More precisely, we prove that

$$-\lim_{n\to\infty} \frac{1}{n} \log \mathrm{P}\Big(\bar{Z}^{(2)}_0 \leq 0, \max_{1 \leq m \leq r} Z^{(1)}_m \leq 0\Big) \approx -\lim_{n\to\infty} \frac{1}{n} \log\Big\{\mathrm{P}(\bar{Z}^{(2)}_0 \leq 0) \prod_{m=1}^{r} \mathrm{P}(Z^{(1)}_m \leq 0)\Big\}$$

and using exchangeability this yields

$$-\lim_{n\to\infty} \frac{1}{n} \log\Big\{\mathrm{P}(\bar{Z}^{(2)}_0 \leq 0)\, \mathrm{P}(Z^{(1)}_0 \leq 0)^r\Big\} = -\lim_{n\to\infty} \frac{1}{n} \log \mathrm{P}(\bar{Z}^{(2)}_0 \leq 0) + rH^{(1)}_k.$$

The second term on the right-hand side is asymptotically r times $1/(2k)$. The first rate is similar to the second rate, except for the additional factor 2 and the replacement of k by r. This gives for the first rate an asymptotic value of $1/(8r)$. Together this yields (21). A similar heuristic argument holds for any s, yielding (17).

We next illustrate the above results using some numerical work. In Table 1, r_k, the value of r which minimizes $H^{(2)}_{k,r}$, is given in terms of k, calculated using standard numerical optimization tools. In Figure 2, $H^{(2)}_{k,r}$ is given for $r = 1, \dots, 5$. It is seen that the values of k where r_k increases are more spread for large k. The asymptotic behaviour shows that those points should be quadratic in k, since $r_k \sim \sqrt{k}/2$ (this follows from $\frac{d}{dr}(\frac{r}{2k} + \frac{1}{8r})|_{r=r_k} = 0$, and is proved in Theorem 5 below).

Figure 3 shows $H^{(s)}_k$ and its asymptotic behaviour for $s = 1, 2, 3$. The results for $s = 3$ are obtained using similar, but more involved methods. It is seen that for $s = 1$, the asymptotic values are very close to the exact (numerical) results. For $s = 2$, the rates also converges quite fast to their asymptotic values, but the convergence is slower than that for $s = 1$. For $s = 3$, there is a significant difference between the exact and the asymptotic result. This is explained by the error term $\mathcal{O}(k^{-1/s})$ (see (17)), which tends to 0 more slowly when s is large.

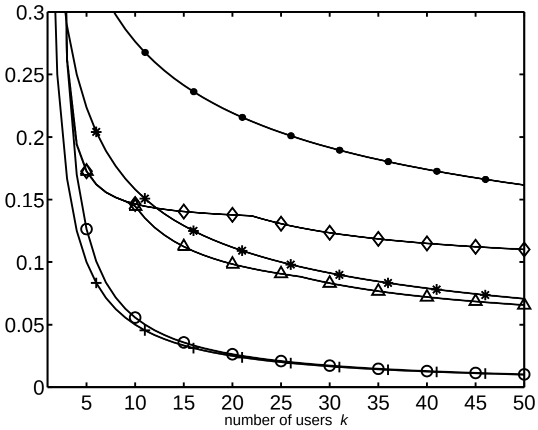

FIGURE 3: Exponential rate H_k^s ($\circ, \triangle, \diamond$) and its asymptotic behaviour $\frac{s}{8}\sqrt[s]{\frac{4}{k}}$ ($+, *, \bullet$) for $s = 1, 2, 3$ respectively.

1.5. Discussion, extensions and open problems

The main result in this paper is Theorem 2. We now discuss a number of its consequences and extensions proved elsewhere. Of course, in practice, one is also interested in letting k and n grow large simultaneously. In [11], Theorem II.5, this case was investigated, and one of the results reads as follows:

Theorem 3. *For all $s \geq 1$, when $k_n \to \infty$ such that $k_n = o(\frac{n}{\log n})$,*

$$\lim_{n\to\infty} -\frac{\sqrt[s]{k_n}}{n} \log \mathrm{P}(\hat{b}_{m1}^{(s)} \neq b_{m1}) \geq \frac{s}{8}\sqrt[s]{4}. \tag{22}$$

The same constant appearing in Theorem 2 appears when $k_n = o(\frac{n}{\log n})$. An important observation in the proof for $s = 2$ is that by the Chernoff inequality, for all r, k and n, and all permutations $m_1, \ldots, m_{k-1}$ of $1, \ldots, k-1$, we have

$$\mathrm{P}\left(\max_{1\leq i\leq r} \mathrm{sgnr}(Z_{m_i}^{(1)}) < 0,\ \min_{i\geq r+1} \mathrm{sgnr}(Z_{m_i}^{(1)}) > 0,\ \mathrm{sgnr}(Z_0^{(2)}) < 0\right) \leq e^{-nH_{k,r}^{(2)}}. \tag{23}$$

Similar bounds hold for $s \geq 3$. The proof is completed by comparing the number of choices of $m_1, \ldots, m_r$ to $e^{-nH_{k,r}^{(2)}}$. This once more explains the practical relevance of the large deviation results presented in this paper.

In a different vein, in [12], the question is addressed when bit-errors appear when k_n is of order $\frac{n}{\log n}$. One of their main results is the following result:

Theorem 4. *When $k_n \to \infty$ such that $k_n = \frac{n}{\gamma \log n}$ with $\gamma < 2$,*

$$\lim_{n\to\infty} \mathrm{P}(\exists 0 \leq j \leq k-1 : \hat{b}_{j1}^{(1)} \neq b_{j1}) = 1. \tag{24}$$

In particular, the MF system needs to be very lightly loaded not to have any bit-errors, which again makes the large deviation results more useful in practice.

In [12], also Theorems 2 and 3 are used to show that when $k_n = \frac{n}{\gamma \log n}$ with $\gamma > \frac{2}{s}$, then the above probability converges to 0. This shows that we can increase the maximal number of users asymptotically by a factor of at least 2 without creating bit-errors when we use at least one stage of HD-PIC. We summarise the above extensions by concluding that the combined results in this paper, in [11] and in [12] shed light on the number of users that the system can maximally allow in the simple MF system, and how this number can be increased by using HD-PIC.

The model that is treated in this paper is rather simplified. We assume that the only noise present is due to the interfering users. Furthermore, we assume that all signals are received with the same power. The same model assumptions on the powers and the absence of other noise sources are used in many of the papers on the approximation of the bit-error probability for the system without interference cancellation, for example [14], [22], [23], [33]. The simple model allows us to investigate the effect of (multi-stage) HD-PIC. However, it is of practical interest to know the behaviour of the system in the case that signals of different users arrive with different powers. Another extension involves additive noise that does not originate from users in the system. It is common to model this with a white noise process. In [19], the exponential rate for the 1-stage HD-PIC system is investigated for the model in which unequal powers and additive noise is taken into account. The result is similar in spirit as Theorem 2 for $s = 2$, and basically shows that the exponential rate when $s = 2$ decays to zero as the square root of the exponential rate for $s = 1$, using similar techniques as presented in this paper.

Another interesting issue is the behaviour of the system when the number of users is fixed and $s \to \infty$. In [11], this question is addressed, and it is shown that when we apply sufficiently many stages of HD-PIC, then the exponential rate becomes at least $\frac{1}{2}\log 2 - \frac{1}{4}$ for *all* $k \geq 2$. Related results are shown when $k \leq \delta n$ with $\delta > 0$ sufficiently small, again illustrating the relevance of using HD-PIC. From a practical point of view, however, it is not clear what is the relevant limiting regime, s fixed and $k \to \infty$ or rather $s \to \infty$ and then $k \to \infty$. This practical issue depends on the number of stages of HD-PIC that can be performed in real-time.

Finally, related scaling results as in Theorem 2 are proved in [6], [7] for a system with an arbitrary number of stages of soft-decision PIC. The proof uses a relation between the eigenvalues of a certain matrix of cross correlations and the performance of SD-PIC. Unfortunately, only a lower bound of the form $J_k^{(s)} \geq \frac{1}{4\sqrt[s]{k}}$ is given, where $J_k^{(s)}$ is the exponential rate of the bit-error probability for the s-stage SD-PIC system.

1.6. Organization of the paper

The paper is organized as follows. In Section 2, we prove the analytical representation for $H_k^{(2)}$ in Theorem 1. In Section 3 we prove the asymptotic behaviour of the exponential rate for $s = 2$ when $k \to \infty$ in Theorem 2. In Section 4-6, we treat the asymptotic behaviour of the exponential rate in Theorem 2 for general s.

2. Exponential rate for $s = 2$: Proof of Theorem 1

In this section we calculate the exponential rate for $s = 2$. In the remainder of this section, we will suppress the indices k, r in $h_{k,r}(\underline{t})$ and write $h(\underline{t}) = h_{k,r}(\underline{t})$.

Proof of Theorem 1. Similarly to [10], we define $\mathcal{X} = \{-1, +1\}^{k-1}$. Furthermore, we define the random vectors $X_i \in \mathcal{X}$ by

$$X_i = A_{0i}(A_{1i}, A_{2i}, \dots, A_{k-1,i})^T, \qquad 1 \leq i \leq n,$$

where the distribution of A_{ji}, for $1 \leq j \leq k$, $1 \leq i \leq n$, is given in (4). It is proven in [10] that the vectors $X_1, \dots, X_n$ are independent and identical distributed. Their common distribution is the uniform distribution on the finite set $\mathcal{X}$, i.e., $\mathrm{P}(X_1 = a) = 2^{1-k}, \quad \forall a \in \mathcal{X}$. Recall (9), to obtain (with $X_{0i} = 1$) that for $s \geq 2$,

$$Z_m^{(s)} = 1 + \sum_{\substack{j=0 \\ j \neq m}}^{k-1} \left(\frac{1}{n} \sum_{i=1}^{n} X_{ji} X_{mi} \right) [1 - \mathrm{sgnr}(Z_j^{(s-1)})],$$

where

$$Z_m^{(1)} = 1 + \sum_{\substack{j=0 \\ j \neq m}}^{k-1} \frac{1}{n} \sum_{i=1}^{n} X_{ji} X_{mi}.$$

We prefer X_{ji} over A_{ji}, because $Z_0^{(s)}$ has the simplified expression

$$Z_0^{(s)} = 1 + \sum_{j=1}^{k-1} \left(\frac{1}{n} \sum_{i=1}^{n} X_{ji} \right) [1 - \mathrm{sgnr}(Z_j^{(s-1)})].$$

Clearly, $1 - \text{sgnr}(\cdot)$ is either 0 or 2. Thus, $Z_0^{(2)}$ has contribution from user m if and only if $\text{sgnr}(Z_m^{(1)}) < 0$. We can intersect the event $\{Z_0^{(2)} \leq 0\}$ by the event $\{\max_{m\in R} \text{sgnr}(Z_m^{(1)}) < 0, \min_{m\in R^c} \text{sgnr}(Z_m^{(1)}) > 0\}$ and average over the sets $R \subseteq \{0,\ldots,k-1\}$. One verifies from exchangeability and (5) that

$$2^{1-k}\sum_{r=1}^{k-1}\binom{k-1}{r}\mathrm{P}\Big(\max_{m\leq r} Z_m^{(1)} \leq 0,\ \min_{m\geq r+1} Z_m^{(1)} \geq 0,\ 1+\frac{2}{n}\sum_{i=1}^{n}\sum_{j=1}^{r} X_{ji} \leq 0\Big) \tag{25}$$

$$\leq \mathrm{P}(Z_0^{(2)} \leq 0) \leq \sum_{r=1}^{k-1}\binom{k-1}{r}\mathrm{P}\Big(\max_{m\leq r} Z_m^{(1)} \leq 0,\ \min_{m\geq r+1} Z_m^{(1)} \geq 0,\ 1+\frac{2}{n}\sum_{i=1}^{n}\sum_{j=1}^{r} X_{ji} \leq 0\Big).$$

Subsequently, we will denote $\bar{Z}_0^{(2)} = 1 + \frac{2}{n}\sum_{i=1}^{n}\sum_{j=1}^{r} X_{ji}$. The bar denotes that we have knowledge of stage 1 and we have replaced the sgnr-functions by its correct value. On an exponential scale, the factor 2^{k-1} will vanish. Thus, it suffices to investigate

$$\sum_{r=1}^{k-1}\binom{k-1}{r}\mathrm{P}\Big(\max_{1\leq m\leq r} Z_m^{(1)} \leq 0,\ \min_{m\geq r+1} Z_m^{(1)} \geq 0,\ 1+\frac{2}{n}\sum_{i=1}^{n}\sum_{j=1}^{r} X_{ji} \leq 0\Big).$$

We note that k is fixed, so that $\lim_{n\to\infty}\frac{1}{n}\log\binom{k-1}{r} = 0$. We next apply the "largest-exponent-wins" principle on the bounds in (25) and find

$$-\lim_{n\to\infty}\frac{1}{n}\log\mathrm{P}(Z_0^{(2)} \leq 0) = \min_{1\leq r\leq k-1} H_{k,r}^{(2)},$$

where

$$H_{k,r}^{(2)} = -\lim_{n\to\infty}\frac{1}{n}\log\mathrm{P}\Big(\max_{1\leq m\leq r} Z_m^{(1)} \leq 0,\ \min_{m\geq r+1} Z_m^{(1)} \geq 0,\ \bar{Z}_0^{(2)} \leq 0\Big).$$

In order to show existence of $H_{k,r}^{(2)}$ and to be able to simplify this expression, we introduce the rate function for $\underline{a} \in \mathbb{R}^r$ and $\underline{b} \in \mathbb{R}^{k-r-1}$,

$$I_r(\underline{a},\underline{b}) = -\lim_{n\to\infty}\frac{1}{n}\log\mathrm{P}\Big(\bigcap_{m=1}^{r}\{Z_m^{(1)} \leq a_m\}\bigcap_{m=r+1}^{k-1}\{Z_m^{(1)} \geq b_m\} \cap \Big\{1+2\sum_{m=1}^{r}\frac{1}{n}\sum_{i=1}^{n} X_{mi} \leq 0\Big\}\Big). \tag{26}$$

Clearly $H_{k,r}^{(2)} = I_r(\underline{0},\underline{0})$, where $\underline{0} = [0,\ldots,0]^T$. Firstly, Cramér's Theorem guarantees existence of this rate, cf. [5], Thm. 2.2.30. This is based on the i.i.d. structure and a finite moment generating function. Secondly, $(\underline{a},\underline{b}) \mapsto I_r(\underline{a},\underline{b})$ is convex, cf. [13], Thm. III.27 and monotone. A third useful fact is exchangeability. Suppose $p_1(\underline{a})$ is a permutation of the elements of $\underline{a}$ and $p_2(\underline{b})$ is a permutation of the elements of $\underline{b}$. Then $I_r(\underline{a},\underline{b}) = I_r(p_1(\underline{a}), p_2(\underline{b}))$, since within a group, the users behave identically. We next prove that

$$I_r(\underline{0},\underline{0}) = -\lim_{n\to\infty}\frac{1}{n}\log\mathrm{P}\Big(\sum_{m=1}^{r} Z_{m+1}^{(1)} \leq 0,\ \sum_{m=r+1}^{k-1} Z_{m+1}^{(1)} \geq 0,\ \bar{Z}_0^{(2)} \leq 0\Big). \tag{27}$$

Indeed, we have that the right-hand side of (27), shortly denoted by RHS equals

$$\min_{\sum a_i\leq 0,\sum b_i\geq 0} I_r(\underline{a},\underline{b})$$

and thus, RHS $\leq I_r(\underline{0},\underline{0})$. To prove the converse we use convexity and exchangeability. Denote the argmin by $\underline{a}^*$ and $\underline{b}^*$ and denote by $\mathcal{P}$ the set of all permutations (p_1,p_2). Then, by exchangeability and convexity respectively

$$I_r(\underline{a}^*,\underline{b}^*) = \frac{1}{|\mathcal{P}|}\sum_{(p_1,p_2)\in\mathcal{P}} I_r(p_1(\underline{a}^*),p_2(\underline{b}^*)) \geq I_r\Big(\frac{1}{|\mathcal{P}|}\sum_{(p_1,p_2)\in\mathcal{P}} p_1(\underline{a}^*),\frac{1}{|\mathcal{P}|}\sum_{(p_1,p_2)\in\mathcal{P}} p_2(\underline{b}^*)\Big). \tag{28}$$

Since $\mathcal{P}$ is the set of all permutations, it is clear that

$$\frac{1}{|\mathcal{P}|}\sum_{(p_1,p_2)\in\mathcal{P}} p_1(\underline{a}^*) = \Big(\sum a_i^*\Big)[1,\dots,1]^T$$

and the same obviously holds for b^*. By monotonicity of I_r and the fact that $\sum a_i^* \le 0$ and $\sum b_i^* \le 0$, we have that the right-hand side of (28) equals

$$I_r\Big(\Big(\sum a_i^*\Big)[1,\dots,1]^T, \Big(\sum b_i^*\Big)[1,\dots,1]^T\Big) \ge I_r(\underline{0},\underline{0})$$

and it follows that RHS $\ge I_r(\underline{0},\underline{0})$. We have now proven (27). The next step is to show that the event $\{\sum_{m\ge r+1} Z_m^{(1)} \ge 0\}$ does not contribute to the rate. Indeed, we use $X_{0i}=1$ and $X_{mi}^2=1$ to obtain

$$\begin{aligned}\sum_{m=1}^{k-1} Z_m^{(1)} &= \frac{1}{n}\sum_{i=1}^{n}\Big(\sum_{m=1}^{k-1}\Big(1+\sum_{\substack{j=0\\ j\ne m}}^{k-1} X_{ji}X_{mi}\Big)\Big) \qquad (29)\\ &= \frac{1}{n}\sum_{i=1}^{n}\Big(\sum_{m=1}^{k-1}\sum_{j=0}^{k-1} X_{ji}X_{mi}\Big) = \frac{1}{n}\sum_{i=1}^{n}\Big(\sum_{j=1}^{k-1} X_{ji} + \Big(\sum_{j=1}^{k-1} X_{ji}\Big)^2\Big) \ge 0\end{aligned}$$

almost surely (a.s.), since $x+x^2 \ge 0$ for $x\in\mathbb{Z}$. Hence, if $\sum_{m=1}^{r} Z_m^{(1)} \le 0$, we necessarily have that $\sum_{m=r+1}^{k-1} Z_m^{(1)} \ge 0$ a.s. This shows that

$$H_{k,r}^{(2)} = -\lim_{n\to\infty}\frac{1}{n}\log \mathrm{P}\Big(\sum_{m=1}^{r} Z_m^{(1)} \le 0,\ \bar{Z}_0^{(2)} \le 0\Big). \qquad (30)$$

We next observe that

$$\begin{bmatrix} \sum_{m=1}^{r} Z_m^{(1)} \\ \bar{Z}_0^{(2)} \end{bmatrix} = \frac{1}{n}\sum_{i=1}^{n}\begin{bmatrix} Y_{1i} \\ Y_{2i}\end{bmatrix},$$

where

$$Y_{1i} = r + \sum_{m=1}^{r}\sum_{\substack{j=0\\ j\ne m}}^{k-1} X_{mi}X_{ji} \quad \text{and} \quad Y_{2i} = 1 + 2\sum_{j=1}^{r} X_{ji}. \qquad (31)$$

Similarly to (29) we obtain

$$Y_{1i} = \Big(\sum_{j=1}^{r} X_{ji}\Big)^2 + \sum_{j=1}^{r} X_{ji}\Big(1+\sum_{j=r+1}^{k-1} X_{ji}\Big), \qquad (32)$$

We abbreviate $Y_i = [Y_{1i}, Y_{2i}]^T$. According to (30), we find that

$$H_{k,r}^{(2)} = -\lim_{n\to\infty}\frac{1}{n}\log \mathrm{P}\Big(\frac{1}{n}\sum_{i=1}^{n}\underline{Y}_i \le \underline{0}\Big),$$

where for a vector $\underline{x}$, we write $\underline{x}\le\underline{0}$ to denote that each entry of $\underline{x}$ is less or equal to 0. Note that $(Y_i)_{i=1}^n$ are i.i.d. and therefore, according to Cramér's Theorem,

$$H_{k,r}^{(2)} = \sup_{\underline{t}\in(-\infty,0]^2}\Big\{-\log \mathrm{E}\, e^{\langle \underline{t},\underline{Y}_1\rangle}\Big\}.$$

In order to show that $\mathrm{E}\, e^{\langle \underline{t},\underline{Y}_1\rangle} = h(\underline{t})$, we condition on $\sum_{l=1}^{r} X_{l1} = j$. Then

$$\Big(\underline{Y}_1\Big|\sum_{l=1}^{r} X_{l1} = j\Big) = \Big[j + j^2 + j\sum_{l=r+1}^{k-1} X_{l1}, 1+2j\Big]^T$$

and thus

$$\mathrm{E}\bigg(e^{\langle \underline{t}, \underline{Y}_1\rangle}\bigg| \sum_{l=1}^{r} X_{l1} = j\bigg) = e^{t_1(j+j^2)+t_2(1+2j)}(\cosh t_1 j)^{k-r-1}.$$

Furthermore, $\mathrm{P}(\sum_{l=1}^{r} X_{l1} = j) = 2^{-r}\binom{r}{\frac{r+j}{2}}$ precisely when $j-r$ is even, so that

$$\mathrm{E}\, e^{\langle \underline{t}, \underline{Y}_1\rangle} = \sum_{j=-r,-r+2}^{r} \mathrm{E}\bigg(e^{\langle \underline{t}, \underline{Y}_1\rangle}\bigg| \sum_{l=1}^{r} X_{l1} = j\bigg)\,\mathrm{P}\bigg(\sum_{l=1}^{r} X_{l1} = j\bigg) = h(\underline{t}).$$

We finally note that $h(\underline{t})$ is log-convex, since it is a moment-generating function.

The next corollary identifies $H^{(2)}_{k,1}$. Unfortunately, we cannot calculate $H^{(2)}_{k,r}$ for $r \geq 2$ explicitly.

Corollary 1. *For all $k \geq 3$,*

$$H^{(2)}_{k,1} = \frac{3}{4}\log 3 - \log 2 + \frac{2k-5}{4}\log\bigg(\frac{2k-5}{2k-4}\bigg) + \frac{2k-3}{4}\log\bigg(\frac{2k-3}{2k-4}\bigg).$$

Proof. For $r = 1$, (16) simplifies to

$$h(\underline{t}) = \frac{1}{2}(\cosh t_1)^{k-2}\Big(e^{-t_2} + e^{2t_1+3t_2}\Big).$$

Setting the partial derivatives equal to zero gives $t_1 = \frac{1}{2}\log\frac{2k-3}{2k-5}$ and $t_2 = \frac{1}{4}\log 3 - \frac{1}{4}\log\frac{2k-3}{2k-5}$. Substitution leads to the desired result.

3. Asymptotic behaviour for $s = 2$: Proof of Theorem 2 for $s = 2$

This section deals with the asymptotic behaviour of $H^{(2)}_{k,r}$, r_k and $H^{(2)}_k$ for $k \to \infty$ in Proposition 1 and Theorem 2, respectively. In Theorem 5, we restate and strengthen the result of Theorem 2 for $s = 2$ by adding the asymptotics of the number of bit-errors in the MF system to obtain a bit-error in the HD-PIC system with $s = 2$. We will first state the results followed by a proof of Proposition 1. The proof of Theorem 5 is deferred to Section 4.

Proposition 1. *For $r \to \infty$ and $\frac{r}{k} \to 0$ as $k \to \infty$,*

$$H^{(2)}_{k,r} = \bigg(\frac{1}{8r} + \frac{r}{2k}\bigg)\bigg(1 + \mathcal{O}\bigg(\frac{1}{8r} + \frac{r}{2k}\bigg)\bigg),\ k \to \infty. \tag{33}$$

Theorem 5. *As $k \to \infty$*

$$H^{(2)}_k = \frac{1}{2\sqrt{k}}\bigg(1 + \mathcal{O}\bigg(\frac{1}{\sqrt{k}}\bigg)\bigg) \qquad \textit{and} \qquad r_k = \frac{1}{2}\sqrt{k} + \mathcal{O}(1).$$

Intuitively the above statement is clear, since $H^{(2)}_k = \min_r H^{(2)}_{k,r}$ and minimizing $\frac{1}{8r} + \frac{r}{2k}$ over r leads to the desired result. The proof is contained in the proof of Theorem 2 in Section 4 and will be omitted here.

Proof of Proposition 1. The proof is divided into 2 steps. In Step 1 we prove the lower bound, in Step 2 the upper bound. We abbreviate $R_0 = \{0,\ldots,k-1\}$, $R_1 = \{1,\ldots,r\}$ and $R_0^+ = R_0 \backslash R_1$. Since $|R_0^+| = k-r$ is used frequently, we abbreviate $k_r = k-r$. For any set $A \subset \mathbb{N} \cup \{0\}$, let

$$S_A = \sum_{j \in A} X_{j1}. \tag{34}$$

In the proof it is useful to observe that for $A \subset \mathbb{N} \cup \{0\}$, we have $\mathrm{E}\, S_A^2 = |A|$ and that there exist constants C_m independent of A such that

$$|\,\mathrm{E}\, S_A^m| \leq C_m |A|^{m/2}. \tag{35}$$

Here and throughout the proof C denotes a strictly positive constant that does not depend on k. C may change from line to line. To obtain the exponential rate $H^{(2)}_{k,r}$ for r large and r/k small, we will not use the expression obtained in Theorem 1. Instead, we start with expression (30). We write $t_1Y_{11} + t_2Y_{21} = Y_q + Y_a$, where (recall (31) and (32))

$$Y_q = t_1\Big(\sum_{j=1}^{r} X_{j1}\Big)^2 + t_2 = t_1 S^2_{R_1} + t_2, \tag{36}$$

$$Y_a = t_1\sum_{j=1}^{r} X_{j1}\left(1 + \sum_{j=r+1}^{k-1} X_{j1}\right) + 2t_2\sum_{j=1}^{r} X_{j1} = t_1 S_{R_1} S_{R_0^+} + 2t_2 S_{R_1}. \tag{37}$$

We shall see that only the first moment of Y_q, representing the part with the *q*uadratic term, will contribute to the rate asymptotically. Furthermore, Y_a (the *a*symmetric part) has mean zero and $\mathrm{E}\,Y_a^3 = \mathrm{E}\,Y_qY_a = \mathrm{E}\,Y_qY_a^3 = 0$. Throughout the entire proof it is sufficient to consider the following moments:

$$\mathrm{E}\,Y_q = t_1 r + t_2, \qquad \mathrm{E}\,Y_a^2 = t_1^2 r k_r + 4t_2^2 r, \tag{38}$$

where we recall that $k_r = k - r$. Using $e^y = 1 + y + y^2e^{\zeta y}/2$ for some $\zeta = \zeta_y \in [0,1]$ and $e^x = 1 + x + x^2/2 + x^3/6 + x^4e^{\eta x}/24$ for some $\eta = \eta_x \in [0,1]$, respectively, we write

$$h(\underline{t}) = \mathrm{E}\,e^{Y_q+Y_a} = 1 + \mathrm{E}\,Y_q + \mathrm{E}\,Y_a^2/2 + e(\underline{t}), \tag{39}$$

where

$$\begin{aligned} e(\underline{t}) &= \mathrm{E}\Big[e^{Y_q}e^{Y_a} - 1 - Y_q - Y_a^2/2\Big] = \mathrm{E}\Big[(1 + Y_q + Y_q^2e^{\zeta Y_q}/2)e^{Y_a} - 1 - Y_q - Y_a^2/2\Big] \\ &= \mathrm{E}\Big[(1+Y_q)(1 + Y_a + Y_a^2/2 + Y_a^3/6 + Y_a^4e^{\eta Y_a}/24) + Y_q^2e^{\zeta Y_q}e^{Y_a}/2 - 1 - Y_q - Y_a^2/2\Big] \\ &= \mathrm{E}\Big[Y_a + Y_a^3/6 + Y_a^4e^{\eta Y_a}/24 + Y_q(Y_a + Y_a^2/2 + Y_a^3/6 + Y_a^4e^{\eta Y_a}/24) + Y_q^2e^{\zeta Y_q}e^{Y_a}\Big]. \end{aligned} \tag{40}$$

Using the symmetry of Y_a, this reduces to

$$e(\underline{t}) = \mathrm{E}\Big[Y_a^4e^{\eta Y_a}/24 + Y_q(Y_a^2/2 + Y_a^4e^{\eta Y_a}/24) + Y_q^2e^{\zeta Y_q}e^{Y_a}\Big]. \tag{41}$$

Step 1: lower bound for $H^{(2)}_{k,r}$. It is sufficient to investigate $h(\underline{t})$ for $\underline{t}^* = (-\frac{1}{k_r}, -\frac{1}{4r})$, since every $\underline{t}$ gives a lower bound for the supremum. Hence, we will substitute $\underline{t}^*$ in (36). We bound, using (41),

$$e(\underline{t}^*) \le \mathrm{E}\Big(Y_a^4e^{|Y_a|}/24 + Y_q^2e^{|Y_a|}\Big),$$

since $\eta Y_a \le |Y_a|$, $Y_q \le 0$ and $e^{\zeta Y_q} \le 1$ a.s. Using Hölder's inequality ([9], Eqn.(7.3.6)) with $p = 3/2$ and $q = 3$ yields

$$e(\underline{t}^*) \le (\mathrm{E}\,Y_a^6)^{2/3}(\mathrm{E}\,e^{3|Y_a|})^{1/3} + (\mathrm{E}\,|Y_q|^3)^{2/3}(\mathrm{E}\,e^{3|Y_a|})^{1/3}. \tag{42}$$

Use (38) to get

$$\mathrm{E}(1 + Y_q + Y_a^2/2) = 1 - \left(\frac{1}{8r} + \frac{r}{2k_r}\right),$$

so that $h(\underline{t}^*) = 1 - (\frac{1}{8r} + \frac{r}{2k_r}) + e(\underline{t}^*)$. It is now sufficient to show that for $\underline{t} = \underline{t}^* = (-\frac{1}{k_r}, -\frac{1}{4r})$, we have that $\mathrm{E}\,e^{3|Y_a|}$ is bounded and that

$$\mathrm{E}\,Y_a^6 = \mathcal{O}\Big(\frac{1}{8r} + \frac{r}{2k_r}\Big)^3 \qquad \text{and} \qquad \mathrm{E}\,|Y_q|^3 = \mathcal{O}\Big(\frac{1}{8r} + \frac{r}{2k_r}\Big)^3.$$

Indeed, then, according to (42), $e(\underline{t}^*) \leq \mathcal{O}\Big(\frac{1}{8r}+\frac{r}{2k_r}\Big)^2$, and it follows that

$$\begin{aligned} H^{(2)}_{k,r} &\geq -\log\bigg(1-\Big(\frac{1}{8r}+\frac{r}{2k_r}\Big)+\mathcal{O}\Big(\frac{1}{8r}+\frac{r}{2k_r}\Big)^2\bigg) \\ &= \Big(\frac{1}{8r}+\frac{r}{2k_r}\Big)+\mathcal{O}\Big(\frac{1}{8r}+\frac{r}{2k_r}\Big)^2=\Big(\frac{1}{8r}+\frac{r}{2k}\Big)+\mathcal{O}\Big(\frac{1}{8r}+\frac{r}{2k}\Big)^2, \end{aligned} \tag{43}$$

which is the desired result. Thus, the remainder of this proof is focused on proving these three statements. Clearly, by symmetry, we have $\mathrm{E}\, e^{3|Y_a|} \leq 2\,\mathrm{E}\, e^{3Y_a}$. Using the Cauchy-Schwarz inequality ([9], Thm. 3.6.9 and Thm.4.6.12) with $X = e^{3t_1^* S_{R_1} S_{R_0^+}}$ and $Y = e^{6t_2^* S_{R_1}}$ results in

$$\mathrm{E}\, e^{3Y_a} \leq \sqrt{\mathrm{E}\, e^{\frac{6}{k_r} S_{R_1} S_{R_0^+}}\ \mathrm{E}\, e^{\frac{3}{r} S_{R_1}}}. \tag{44}$$

In order to prove that the expression above is bounded, the following lemma will be useful.

Lemma 1. *Suppose $A_1, A_2 \subset \mathbb{N}\cup\{0\}$ are disjoint. Let $S_A = \sum_{m\in A} X_{m1}$. Then $\mathrm{E}\, e^{\frac{x}{|A_1|} S_{A_1} S_{A_2}}$ is uniformly bounded whenever $\frac{x^2|A_2|}{|A_1|} \leq 1-\varepsilon$, $0<\varepsilon<1$ fixed.*

Proof. Since A_1 and A_2 are disjoint, S_{A_1} and S_{A_2} are independent and $\{0\}\in A_1$ or $\{0\} \in A_2$ but not both. Suppose $\{0\} \notin A_2$. Then, using $\cosh x \leq e^{x^2/2}$,

$$\mathrm{E}\, e^{\frac{x}{|A_1|} S_{A_1} S_{A_2}} = \mathrm{E}\bigg(\cosh\Big(\frac{x}{|A_1|} S_{A_1}\Big)\bigg)^{|A_2|} \leq \mathrm{E}\, e^{\frac{x^2|A_2|}{2|A_1|}\frac{S_{A_1}^2}{|A_1|}} \leq \mathrm{E}\, e^{\frac{1-\varepsilon}{2}\frac{S_{A_1}^2}{|A_1|}}.$$

When $\{0\} \in A_2$, we apply $\cosh x \leq e^{x^2/2}$ on S_{A_1}, resulting in the same expression with A_1 replaced by A_2. Note that $(1+S_A)^2 \stackrel{d}{=} (1-S_A)^2 = (-1+S_A)^2$, so that we may assume that $\{0\} \notin A_1$. Finally, since for $Z \sim \mathcal{N}(0,1)$, $\mathrm{E}\, e^{tZ} = e^{t^2/2}$,

$$\mathrm{E}\, e^{\frac{1-\varepsilon}{2}\frac{S_{A_1}^2}{|A_1|}} = \mathrm{E}\, e^{\sqrt{1-\varepsilon}\frac{S_{A_1}}{\sqrt{|A_1|}} Z} = \mathrm{E}\prod_{i\in A_1}\cosh\Big(\sqrt{1-\varepsilon}\frac{Z}{\sqrt{|A_1|}}\Big) \leq \mathrm{E}\, e^{\frac{1-\varepsilon}{2} Z^2} = \frac{1}{\sqrt{\varepsilon}} < \infty. \tag{45}$$

Since $|R_0^+| = k_r$ and $|R_1| = r$ we can apply Lemma 1 (note that both r/k_r and $1/r$ are $o(1)$ and thus clearly $\leq 1-\varepsilon$ when k is sufficiently large), so that indeed $\mathrm{E}\, e^{3|Y_a|}$ is bounded. Using $(x+y)^n \leq 2^{n-1}(x^n+y^n)$, it is straightforward to show that for $\underline{t}^* = (-\frac{1}{k_r}, -\frac{1}{4r})$,

$$\mathrm{E}\, Y_a^6 \leq C\bigg(\frac{1}{r^6}\mathrm{E}\, S_{R_1}^6 + \frac{1}{k_r^6}\mathrm{E}\, S_{R_1}^6\,\mathrm{E}\, S_{R_0^+}^6\bigg).$$

We obtain, using (35) and $|R_1| = r$, $|R_0^+| = k_r$,

$$\mathrm{E}\, Y_a^6 \leq C\Big(\frac{1}{r^3}+\frac{r^3}{k_r^3}\Big) \leq C\Big(\frac{1}{8r}+\frac{r}{2k_r}\Big)^3.$$

Similarly,

$$\mathrm{E}\,|Y_q|^3 \leq C\Big(\frac{1}{r^3}+\frac{1}{k_r^3}\mathrm{E}\, S_{R_1}^6\Big) \leq C\Big(\frac{1}{r^3}+\frac{r^3}{k_r^3}\Big) \leq C\Big(\frac{1}{8r}+\frac{r}{2k_r}\Big)^3.$$

This completes the proof of the lower bound for $H^{(2)}_{k,r}$.

Step 2: upper bound for $H^{(2)}_{k,r}$. We will define an appropriate ellipse $\mathcal{E}$ with $\underline{0} \in \mathcal{E}^0$, the interior of $\mathcal{E}$. In order to show that the supremum of $-\log h(\underline{t})$ is attained in $\mathcal{E}^0$, it is sufficient to show that on the boundary of the ellipse $h(\underline{t}) > 1$. Since $h(\underline{0}) = 1$ and h is log-convex, we can then conclude that $h(\underline{t}) > 1$ outside the ellipse and thus the supremum is never attained there. Indeed, whenever $\underline{t} \notin \mathcal{E}$, there exists a unique $0<\alpha<1$ such that $\alpha\underline{t} \in \partial\mathcal{E}$. From convexity of h and $h(\alpha\underline{t}) > 1$ it follows that

$$1 < h(\alpha\underline{t}) = h(\alpha\underline{t}+(1-\alpha)\underline{0}) \leq \alpha h(\underline{t}) + (1-\alpha)h(\underline{0}) = \alpha h(\underline{t}) + (1-\alpha).$$

It immediately follows that $h(\underline{t}) > 1$. Whenever $\underline{t} \in \mathcal{E}^0$, we can prove the desired upper bound. We often minimize $h(\underline{t})$, rather than maximizing $-\log h(\underline{t})$. Note that we have

$$h(\underline{t}) \geq 1 + \mathrm{E}\, Y_q + \mathrm{E}\, Y_a^2/2 + e(\underline{t}),$$

where according to (41), $e(\underline{t}) \geq \mathrm{E}\, Y_q(Y_a^2/2 + Y_a^4 e^{\zeta Y_a}/24)$. Substitution of the moments in (38) leads to

$$\begin{aligned} h(\underline{t}) &\geq 1 + t_1 r + t_2 + t_1^2 r k_r/2 + 2t_2^2 r + \mathrm{E}\, Y_q Y_a^2/2 \qquad (46)\\ &= 1 - \left(\frac{1}{8r} + \frac{r}{2k_r}\right) + \frac{1}{2} r k_r \left(t_1 + \frac{1}{k_r}\right)^2 + 2r\left(t_2 + \frac{1}{4r}\right)^2 + e(\underline{t}). \end{aligned}$$

Define

$$\mathcal{E} = \left\{\underline{t} : \frac{1}{2} r k_r \left(t_1 + \frac{1}{k_r}\right)^2 + 2r\left(t_2 + \frac{1}{4r}\right)^2 \leq 2\left(\frac{1}{8r} + \frac{r}{2k_r}\right)\right\}.$$

Note that the ellipse indeed contains $\underline{t} = \underline{0}$. For $\underline{t} \in \partial\mathcal{E}$ we have by the triangle inequality

$$|t_1| \leq C(\frac{1}{8r} + \frac{r}{2k_r})^{1/2}/(rk_r)^{1/2}, \qquad |t_2| \leq C(\frac{1}{8r} + \frac{r}{2k_r})^{1/2}/r^{1/2}.$$

Concerning $e(\underline{t})$, we use the Cauchy-Schwarz inequality to bound $\mathrm{E}\, Y_q Y_a^2$ for $\underline{t} \in \mathcal{E}$,

$$|\,\mathrm{E}\, Y_q Y_a^2| \leq (\mathrm{E}\, Y_q^2)^{1/2} (\mathrm{E}\, Y_a^4)^{1/2}.$$

For convenience we further use Hölder's inequality in the form $\mathrm{E}\,|Z|^p \leq (\mathrm{E}\,|Z|^q)^{p/q}$ for $p \leq q$ and any random variable Z to obtain[1]

$$|\,\mathrm{E}\, Y_q Y_a^2| \leq (\mathrm{E}\, Y_q^4)^{1/4} (\mathrm{E}\, Y_a^6)^{2/3}. \qquad (47)$$

Using $(x+y)^n \leq 2^{n-1}(x^n + y^n)$, we arrive for $\underline{t} \in \mathcal{E}$ at

$$\begin{aligned} \mathrm{E}\, Y_q^4 &\leq C(t_1^4\, \mathrm{E}\, S_{R_1}^8 + t_2^4) \leq C(t_1^4 r^4 + t_2^4) \leq C\left(\frac{(\frac{1}{8r} + \frac{r}{2k_r})^2}{r^2 k_r^2} r^4 + \frac{(\frac{1}{8r} + \frac{r}{2k_r})^2}{r^2}\right) \qquad (48)\\ &\leq C\left(\frac{1}{8r} + \frac{r}{2k_r}\right)^4, \\ \mathrm{E}\, Y_a^6 &\leq C(t_1^6\, \mathrm{E}\, S_{R_0^+}^6\, \mathrm{E}\, S_{R_1}^6 + t_2^6\, \mathrm{E}\, S_{R_1}^6) \leq C(t_1^6 r^3 k_r^3 + t_2^6 r^3) \qquad (49)\\ &\leq C\left(\frac{(\frac{1}{8r} + \frac{r}{2k_r})^3}{r^3 k_r^3} r^3 k_r^3 + \frac{(\frac{1}{8r} + \frac{r}{2k_r})^3}{r^3} r^3\right) \leq C\left(\frac{1}{8r} + \frac{r}{2k_r}\right)^3. \end{aligned}$$

This implies $|\,\mathrm{E}\, Y_q Y_a^2| \leq C\left(\frac{1}{8r} + \frac{r}{2k_r}\right)^2$.

For the second term in $e(\underline{t})$, we use $\zeta Y_a \leq |Y_a|$ a.s. (since $|\xi| \leq 1$), and Holdër's inequality twice to obtain

$$|\,\mathrm{E}\, Y_q Y_a^4 e^{\zeta Y_a}| \leq (\mathrm{E}\, Y_a^4)^{1/4} (\mathrm{E}\, Y_a^6)^{2/3} (\mathrm{E}\, e^{12|Y_a|})^{1/12}.$$

The product of the first two expectations on the right-hand side are bounded by $(\frac{1}{8r} + \frac{r}{2k_r})^2$, by the bounds (48) and (49), so it suffices to show that $\mathrm{E}\, e^{12|Y_a|}$ is bounded. Following the approach in the lower bound (see (44)), this fact follows from Cauchy-Schwarz and Lemma 1, together with the bounds on t_1 and t_2. We therefore conclude that for all $\underline{t} \in \partial\mathcal{E}$ and k sufficiently large

$$h(\underline{t}) \geq 1 + \left(\frac{1}{8r} + \frac{r}{2k_r}\right)\left(-1 + 2 - C\left(\frac{1}{8r} + \frac{r}{2k_r}\right)\right) > 1$$

and thus we can conclude that the infimum over $h(\underline{t})$ is not attained on the complement of the ellipse. From (46), we know that for $\underline{t} \in \mathcal{E}^0$,

$$h(\underline{t}) \geq 1 - \frac{1}{8r} - \frac{r}{2k_r} + \frac{1}{2} r k_r \left(t_1 + \frac{1}{k_r}\right)^2 + 2r\left(t_2 + \frac{1}{4r}\right)^2 - C\left(\frac{1}{8r} + \frac{r}{2k_r}\right)^2.$$

It is clear that the minimum of the right-hand side is attained at $\underline{t} = (-\frac{1}{k_r}, -\frac{1}{4r})$ and this leads to

$$\inf_{\underline{t}} h(\underline{t}) \geq 1 - \left(\frac{1}{8r} + \frac{r}{2k_r}\right)\left(1 + \mathcal{O}\left(\frac{1}{8r} + \frac{r}{2k_r}\right)\right).$$

Finally, the derivation in (43) completes the proof of Proposition 1.

4. Asymptotic behaviour for general s

In Section 3, we have demonstrated a technique to deal with the asympotic behaviour of $H_k^{(2)}$, using Taylor series expansions of moment generating functions and an ellipse argument to deal with the arising minimization problem. In this section, we will show that the same technique enables us to investigate multistage HD-PIC. Of course, for $s > 2$, the analysis becomes more involved. The main result is Theorem 2, an extension of Theorem 5 for general fixed s. The proof is based on Proposition 2, which is the extension of Proposition 1. We will need some additional notation. Define for $1 \leq \sigma \leq s-1$, $R_\sigma = \{m : \mathrm{sgnr}_m(Z_m^{(\sigma)}) < 0\}$. We take by definition $R_0 = \{0, \ldots, k-1\}$, $R_s = \{0\}$ and $R_{s+1} = \emptyset$, the empty set. For $s = 2$, we have $R_1 = \{1, \ldots, r\}$. Similarly to $s = 2$, we will investigate the situation where we include the configuration of signs of $(Z_j^{(\sigma)})_{j=0}^k$ denoted by $\underline{R} = [R_1, \ldots, R_{s-1}]$. Thus, we introduce

$$H_{k,\underline{R}}^{(s)} = -\lim_{n\to\infty} \frac{1}{n} \log \mathrm{P}\Bigg(\bigcap_{1\leq\sigma\leq s} \left\{ \max_{m\in R_\sigma} \mathrm{sgnr}_m(Z_m^{(\sigma)}) < 0 \right\} \bigcap_{1\leq\sigma\leq s-1} \left\{ \min_{m\in R_\sigma^c} \mathrm{sgnr}_m(Z_m^{(\sigma)}) > 0 \right\} \Bigg). \tag{50}$$

Similarly to the case $s = 2$, we then have $H_k^{(s)} = \min_{\underline{R}} H_{k,\underline{R}}^{(s)}$. In particular, this shows that the exponential rate of the bit-error probability $H_k^{(s)}$ in (11) exists. We have proven in the previous section that $H_k^{(2)} = (\frac{1}{8r} + \frac{r}{2k})(1 + \mathcal{O}(\frac{1}{8r} + \frac{r}{2k}))$. The analog of $\frac{1}{8r} + \frac{r}{2k}$ for general s is given by

$$\mathcal{H} = \frac{1}{2} \sum_{\sigma=1}^{s} 4^{-\mathbb{1}_{\{\sigma\geq 2\}}} \frac{|R_\sigma|}{|R_{\sigma-1}|}. \tag{51}$$

The following result is the key ingredient to the asymptotics of $H_k^{(s)}$. The proof is deferred to Section 5.

Proposition 2. *Fix $1 \leq s < \infty$. When $\mathcal{H} = o(1)$ as $k \to \infty$, the two following properties for $H_{k,\underline{R}}^{(s)}$ hold:*
(i)

$$H_{k,\underline{R}}^{(s)} \geq \mathcal{H}(1 + \mathcal{O}(\mathcal{H})).$$

(ii) When $R_\sigma = \{\sum_{\sigma'=\sigma+1}^s |R_{\sigma'}| + 1, \ldots, \sum_{\sigma'=\sigma}^s |R_{\sigma'}|\}$ for all $1 \leq \sigma \leq s-1$, then

$$H_{k,\underline{R}}^{(s)} \leq \mathcal{H}(1 + \mathcal{O}(\mathcal{H})).$$

Remark: For $s = 2$, the additional condition in (ii) follows from exchangeability.

Proposition 2 is the main ingredient to the asymptotics of $H_k^{(s)}$. However, we need the following additional fact:

Corollary 2. *Fix $\alpha > 0$. Let $A_1, A_2 \subset \mathbb{N} \cup \{0\}$. When $|A_1|/|A_2| \to 0$,*

$$-\lim_{n\to\infty} \frac{1}{n} \log \mathrm{P}\left(\max_{m\in A_1} \left\{ \alpha + \frac{1}{n} \sum_{i=1}^{n} \sum_{j\in A_2\setminus\{m\}} A_{ji}A_{mi} \right\} \leq 0 \right) \geq \frac{\alpha^2|A_1|}{2|A_2|}\left(1 + \mathcal{O}\left(\frac{|A_1|}{|A_2|}\right)\right).$$

The proof is similar to the proof of Proposition 2 and is sketched in Section 6.

Proof of Theorem 33. Since $H_k^{(s)} = \min_{\underline{R}} H_{k,\underline{R}}^{(s)}$, substituting a specific configuration $(R_\sigma)_{\sigma=1}^s$ leads to an upper bound of $H_k^{(s)}$. We substitute the hierarchical configuration with $|R_\sigma| =$

$\lceil (k/4)^{(s-\sigma)/s} \rceil = (k/4)^{(s-\sigma)/s}(1+\mathcal{O}(k^{-(s-\sigma)/s}))$ in $\mathcal{H}(1+\mathcal{O}(\mathcal{H}))$ (recall Proposition 2 (ii)) to obtain

$$H_k^{(s)} \leq \frac{s\sqrt[s]{4}}{8\sqrt[s]{k}}\left(1+\mathcal{O}\left(\frac{1}{\sqrt[s]{k}}\right)\right).$$

Thus, it is sufficient to prove that

$$H_k^{(s)} = \min_{\underline{R}} H_{k,\underline{R}}^{(s)} \geq \min_{\underline{R}} \mathcal{H}(1+\mathcal{O}(\mathcal{H})) \geq \frac{s\sqrt[s]{4}}{8\sqrt[s]{k}}\left(1+\mathcal{O}\left(\frac{1}{\sqrt[s]{k}}\right)\right). \tag{52}$$

The first inequality follows from Proposition 2 (i). So let us investigate

$$\min_{\underline{R}} \mathcal{H}(1+\mathcal{O}(\mathcal{H})) = \min_{R_1,R_2,\ldots,R_{s-1}} \mathcal{H}(1+\mathcal{O}(\mathcal{H})).$$

Since $\mathcal{H}$ is a function only of the cardinalities of R_σ, this equals

$$\min_{|R_1|\in\mathbb{N},\ldots,|R_{s-1}|\in\mathbb{N}} \mathcal{H}(1+\mathcal{O}(\mathcal{H})).$$

We replace the condition that $|R_\sigma| \in \mathbb{N}$ by $|R_\sigma| \geq 1$; this will result in a lower bound of $H_k^{(s)}$. It has the main advantage that it allows us to differentiate with respect to $|R_\sigma|$. Let us first assume that $\frac{|R_\sigma|}{|R_{\sigma-1}|} = o(1)$ for all $1 \leq \sigma \leq s$. When k is sufficiently large, there exists an $M \geq 0$, such that $H_{k,\underline{R}}^{(s)} \geq \mathcal{H} - M\mathcal{H}^2$. Putting the partial derivatives of the lower bound with respect to $|R_\sigma|$ equal to zero leads to

$$\frac{1}{8}\left(\frac{4^{\mathbb{1}_{\{\sigma=1\}}}}{|R_{\sigma-1}|} - \frac{|R_{\sigma+1}|}{|R_\sigma|^2}\right)\Big(1-2M\mathcal{H}\Big) = 0, \text{ for all } \sigma = 1,\ldots,s.$$

Since $\mathcal{H} = o(1)$ it follows that when k is sufficiently large, the condition for a minimum is

$$\left(\frac{4^{\mathbb{1}_{\{\sigma=1\}}}}{|R_{\sigma-1}|} - \frac{|R_{\sigma+1}|}{|R_\sigma|^2}\right) = 0, \quad \text{for all } 1 \leq \sigma \leq s,$$

leading to $|R_\sigma| = (k/4)^{(s-\sigma)/s}$. Substitution yields (52). We note that we are allowed to substitute $|R_\sigma| = (k/4)^{(s-\sigma)/s}(1+\mathcal{O}(k^{-1/s}))$ without changing the order of the error terms. This proves for example that for $s=2$ we have $r_k = \frac{1}{2}\sqrt{k} + \mathcal{O}(1)$. We finally need to rule out the possibility that $\frac{|R_\sigma|}{|R_{\sigma-1}|}$ is not $o(1)$ for some $1 \leq \sigma = \sigma^* \leq s$. Clearly $|R_{\sigma^*-1}| \to \infty$; otherwise the rate is strictly positive uniformly in k. Since $\mathrm{P}(A \leq 0, B \leq 0) \leq \mathrm{P}(A \leq 0)$, the rate of the event

$$\bigcap_{1\leq\sigma\leq s}\left\{\max_{m\in R_\sigma} \mathrm{sgnr}(Z_m^{(\sigma)}) < 0\right\} \bigcap_{1\leq\sigma\leq s}\left\{\min_{m\in R_\sigma^c} \mathrm{sgnr}(Z_m^{(\sigma)}) > 0\right\}$$

is bounded from below by

$$\lim_{n\to\infty} -\frac{1}{n}\log \mathrm{P}\left(\max_{m\in R_{\sigma^*}}\left\{\frac{1}{n}\sum_{i=1}^n\left(1+2^{\mathbb{1}_{\{\sigma^*\geq 2\}}}\sum_{\substack{j\in R_{\sigma^*-1}\\ j\neq m}} X_{ji}X_{mi}\right)\right\} \leq 0\right).$$

The event $\{\max_{m\in R_{\sigma^*}}\{\ldots\} \leq 0\}$ is clearly increasing in $|R_{\sigma^*}|$. Replacing R_{σ^*} by an $R' \subset R_{\sigma^*}$ will result in a decrease of the above rate. Taking $|R'| = \lceil 4s\sqrt[s]{4}|R_{\sigma-1}|/\sqrt[s]{k}\rceil$ and using Corollary 2 gives

$$\begin{aligned}
&\lim_{n\to\infty} -\frac{1}{n}\log \mathrm{P}\left(\max_{m\in R_{\sigma^*}}\left\{\frac{1}{n}\sum_{i=1}^n 1+2^{\mathbb{1}_{\{\sigma^*\geq 2\}}}\sum_{\substack{j\in R_{\sigma^*-1}\\ j\neq m}} X_{ji}X_{mi}\right\} \leq 0\right) \\
&\geq \frac{|R'|}{8|R_{\sigma^*-1}|}(1+o(1)) \geq \frac{s\sqrt[s]{4}}{2\sqrt[s]{k}}(1+o(1)) > \frac{s\sqrt[s]{4}}{8\sqrt[s]{k}}\left(1+\mathcal{O}\left(\frac{1}{\sqrt[s]{k}}\right)\right),
\end{aligned}$$

when k is sufficiently large. In other words, when $\frac{|R_\sigma|}{|R_{\sigma-1}|}$ is not $o(1)$ for some $1 \leq \sigma = \sigma^* \leq s$, the minimum is not attained. This completes the proof of Theorem 2.

5. Proof of Proposition 2

We shall prove Proposition 2 as much as possible along the same lines as Proposition 1. Substituting $\mathrm{sgnr}_m(Z_m^{(\sigma)})$ in (50) and using similar bounds as in (25) yields

$$H_{k,\underline{R}}^{(s)} = -\lim_{n\to\infty}\frac{1}{n}\log \mathrm{P}\Bigg(\bigcap_{\sigma=1}^{s}\Big\{\max_{m\in R_\sigma}\bar{Z}_m^{(\sigma)}\le 0\Big\}\bigcap_{\sigma=1}^{s-1}\Big\{\min_{m\in R_\sigma^c}\bar{Z}_m^{(\sigma)}\ge 0\Big\}\Bigg), \tag{53}$$

where

$$\bar{Z}_m^{(\sigma)} = 1 + 2^{\mathbb{1}_{\{\sigma\ge 2\}}}\sum_{j\in R_{\sigma-1}\setminus\{m\}}\frac{1}{n}\sum_{i=1}^{n}X_{mi}X_{ji}.$$

Step 1*: Proof of the lower bound* (*i*). Since we deal with a lower bound, we are allowed to omit certain events. More precisely, in (53), we discard the events $\{\cdot\ge 0\}$ and we replace $\max_{m\in R_\sigma}$ by $\sum_{m\in R_\sigma}$ (because $\mathrm{P}(\max_{m\in R_\sigma}\cdot\le 0)\le \mathrm{P}(\sum_{m\in R_\sigma}\cdot\le 0)$) to obtain

$$H_{k,\underline{R}}^{(s)} \ge -\lim_{n\to\infty}\frac{1}{n}\log \mathrm{P}\Bigg(\bigcap_{\sigma=1}^{s}\Big\{\sum_{m\in R_\sigma}\bar{Z}_m^{(\sigma)}\le 0\Big\}\Bigg).$$

We write this as (compare (30)-(32))

$$H_{k,\underline{R}}^{(s)} \ge -\lim_{n\to\infty}\frac{1}{n}\log \mathrm{P}\Bigg(\frac{1}{n}\sum_{i=1}^{n}\underline{Y}_i\le \underline{0}\Bigg),$$

where $\underline{Y}_i$ is an i.i.d. random vector with coordinates

$$\begin{aligned} Y_{\sigma,i} &= \sum_{m\in R_\sigma}\Big\{1+2^{\mathbb{1}_{\{\sigma\ge 2\}}}\sum_{j\in R_{\sigma-1}\setminus\{m\}}X_{mi}X_{ji}\Big\}\\ &= |R_\sigma| - 2^{\mathbb{1}_{\{\sigma\ge 2\}}}|R_\sigma\cap R_{\sigma-1}| + 2^{\mathbb{1}_{\{\sigma\ge 2\}}}\sum_{m\in R_\sigma}X_{mi}\sum_{j\in R_{\sigma-1}}X_{ji}\end{aligned}$$

and where for a vector $\underline{x}$, we write $\underline{x}\le\underline{0}$ to denote that each entry of $\underline{x}$ is less than or equal to zero. In the last equality, we have used that $\sum_{j\in R_{\sigma-1}\setminus\{m\}}X_{ji}X_{mi} = \sum_{j\in R_{\sigma-1}}X_{ji}X_{mi} - \mathbb{1}_{\{m\in R_{\sigma-1}\}}$ and $\sum_{m\in R_\sigma}\mathbb{1}_{\{m\in R_{\sigma-1}\}} = |R_\sigma\cap R_{\sigma-1}|$. Cramér's Theorem gives

$$H_{k,\underline{R}}^{(s)} \ge \sup_{\underline{t}\le\underline{0}}\{-\log h(\underline{t})\},$$

where $h(\underline{t}) = \mathrm{E}\,e^{<\underline{t},\underline{Y}_1>}$ is the moment generating function of $\underline{Y}_1$. The area $\underline{t}\le\underline{0}$ naturally arises since we are dealing with events of the form $\{\cdot\le 0\}$, which implies that all t_i's are non-positive. We often prefer to minimize $h(\underline{t})$, instead of maximizing $-\log h(\underline{t})$. We invoke the notation $S_A=\sum_{j\in A}X_{j1}$, which we have introduced in (34). Note that $R_\sigma = R_\sigma^-\cup(R_\sigma\cap R_{\sigma-1})$ and that $R_{\sigma-1}=R_{\sigma-1}^+\cup(R_\sigma\cap R_{\sigma-1})$, where $R_\sigma^+ = R_\sigma\setminus R_{\sigma+1}$ and $R_\sigma^- = R_\sigma\setminus R_{\sigma-1}$. Recall that for $s=2$, we have $R_0=\{0,\ldots,k-1\}$ and $R_1=\{1,\ldots,r\}$, so that $R_0^+=\{0,r+1,\ldots,k-1\}$ and $R_1^-=\emptyset$. For convenience, we split the inner product $\langle\underline{t},\underline{Y}_1\rangle = Y_q+Y_a$, where we use the expressions for R_σ and $R_{\sigma-1}$ above and define:

$$\begin{aligned} Y_q &= \sum_{\sigma=1}^{s}t_\sigma\Bigg[|R_\sigma| - 2^{\mathbb{1}_{\{\sigma\ge 2\}}}|R_\sigma\cap R_{\sigma-1}| + 2^{\mathbb{1}_{\{\sigma\ge 2\}}}\Bigg(\sum_{j\in R_\sigma\cap R_{\sigma-1}}X_{j1}\Bigg)^2\Bigg]\\ &= \sum_{\sigma=1}^{s}t_\sigma\Big[|R_\sigma| - 2^{\mathbb{1}_{\{\sigma\ge 2\}}}|R_\sigma\cap R_{\sigma-1}| + 2^{\mathbb{1}_{\{\sigma\ge 2\}}}S_{R_\sigma\cap R_{\sigma-1}}^2\Big],\\ Y_a &= \sum_{\sigma=1}^{s}t_\sigma 2^{\mathbb{1}_{\{\sigma\ge 2\}}}\Bigg[\Bigg(\sum_{j\in R_\sigma^-}X_{j1}\Bigg)\Bigg(\sum_{j\in R_{\sigma-1}^+}X_{j1}\Bigg) + \Bigg(\sum_{j\in R_{\sigma-1}\cap R_\sigma}X_{j1}\Bigg)\Bigg(\sum_{j\in R_\sigma^-\cup R_{\sigma-1}^+}X_{j1}\Bigg)\Bigg]\\ &= \sum_{\sigma=1}^{s}t_\sigma 2^{\mathbb{1}_{\{\sigma\ge 2\}}}\Big[S_{R_\sigma^-}S_{R_{\sigma-1}^+} + S_{R_{\sigma-1}\cap R_\sigma}S_{R_\sigma^-\cup R_{\sigma-1}^+}\Big].\end{aligned}$$

Similarly to $s = 2$, we have $\mathrm{E}\, Y_a = 0$. However, for general s, $\mathrm{E}\, Y_a e^{Y_q} \neq 0$, $\mathrm{E}\, Y_a^3 e^{Y_q} \neq 0$, so that we have to deal with additional error terms. We substitute $t^*_\sigma = -4^{-\mathbb{1}_{\{\sigma \geq 2\}}}/|R_{\sigma-1}|$, which will lead to a lower bound of the rate. We will prove that $h(\underline{t}^*) \leq 1 - \mathcal{H} + \mathcal{O}(\mathcal{H}^2)$. Since $-\log(1 - x + \mathcal{O}(x^2)) = x(1 + \mathcal{O}(x))$, the claim in (i) then follows. We have (recall (39))

$$h(\underline{t}^*) = 1 + \mathrm{E}\, Y_q + \frac{1}{2}\mathrm{E}\, Y_a^2 + e(\underline{t}^*),$$

where $e(\underline{t}^*)$ is given by the right-hand side of (40). Similarly to (42) we use $\zeta Y_a \leq |Y_a|$, Hölder's inequality and the fact that $\mathrm{E}\,|Z|^p \leq (\mathrm{E}\,|Z|^q)^{p/q}$ for $p \leq q$ and any random variable Z to bound $e(t^*)$. Using this gives that $e(\underline{t})$ is bounded from above by

$$\mathrm{E}\, Y_a^3 + \Big(\mathrm{E}\, Y_a^6\Big)^{2/3}\Big(\mathrm{E}\, e^{3|Y_a|}\Big)^{1/3} + \mathrm{E}\, Y_q Y_a + \Big(\mathrm{E}\, Y_q^4\Big)^{1/4}\Big(\mathrm{E}\, Y_a^6\Big)^{1/3} \tag{54}$$
$$+\Big(\mathrm{E}\, Y_q^4\Big)^{1/4}\Big(\mathrm{E}\, Y_a^6\Big)^{1/2} + \Big(\mathrm{E}\, Y_q^4\Big)^{1/4}\Big(\mathrm{E}\, Y_a^6\Big)^{2/3}\Big(\mathrm{E}\, e^{12|Y_a|}\Big)^{1/12} + \Big(\mathrm{E}\, Y_q^4 e^{2\zeta Y_q}\Big)^{1/2}\Big(\mathrm{E}\, e^{2Y_a}\Big)^{1/2}.$$

Since $Y_q \leq \mathcal{H}$ a.s. and $\mathcal{H} \to 0$, we have $\mathrm{E}\,|Z|e^{3/2\zeta Y_q} \leq 2\,\mathrm{E}\,|Z|$ and $\mathrm{E}\,|Z|e^{2\zeta Y_q} \leq 2\,\mathrm{E}\,|Z|$ for any random variable Z when k is sufficiently large. As a result we can discard the factor $e^{2\zeta Y_q}$ in (54). We will prove that $\mathrm{E}\, Y_q$ and $\mathrm{E}\, Y_a^2$ are the main contributions to the rate and that $e(\underline{t}^*)$ is of lower order. We have

$$\begin{aligned}\mathrm{E}\, Y_q &= \sum_{\sigma=1}^s t^*_\sigma\Big[|R_\sigma| - 2^{\mathbb{1}_{\{\sigma\geq 2\}}}|R_{\sigma-1}\cap R_\sigma| + 2^{\mathbb{1}_{\{\sigma\geq 2\}}}\,\mathrm{E}\, S^2_{R_{\sigma-1}\cap R_\sigma}\Big]\\ &= \sum_{\sigma=1}^s t^*_\sigma |R_\sigma| = -\sum_{\sigma=1}^s 4^{-\mathbb{1}_{\{\sigma\geq 2\}}}\frac{|R_\sigma|}{|R_{\sigma-1}|} = -2\mathcal{H},\end{aligned}$$

Moreover,

$$\frac{1}{2}\mathrm{E}\, Y_a^2 = \frac{1}{2}\sum_{\sigma=1}^s 4^{\mathbb{1}_{\{\sigma\geq 2\}}}(t^*_\sigma)^2\,\mathrm{E}\Big[S_{R^-_\sigma}S_{R^+_{\sigma-1}} + S_{R_{\sigma-1}\cap R_\sigma}S_{R^-_\sigma\cup R^+_{\sigma-1}}\Big]^2 + e_1(\underline{t}^*),$$

where

$$\begin{aligned}e_1(\underline{t}^*) = &\sum_{1\leq\sigma<\sigma'\leq s} t^*_\sigma t^*_{\sigma'} 2^{\mathbb{1}_{\{\sigma\geq 2\}}+\mathbb{1}_{\{\sigma'\geq 2\}}}\\ &\times \mathrm{E}\Big\{\Big[S_{R^-_\sigma}S_{R^+_{\sigma-1}} + S_{R_{\sigma-1}\cap R_\sigma}S_{R^+_{\sigma-1}\cup R^-_\sigma}\Big]\Big[S_{R^-_{\sigma'}}S_{R^+_{\sigma'-1}} + S_{R_{\sigma'-1}\cap R_{\sigma'}}S_{R^-_{\sigma'}\cup R^+_{\sigma'-1}}\Big]\Big\}.\end{aligned}$$

A straightforward calculation gives that

$$\mathrm{E}\, S_{R^-_\sigma}S_{R^+_{\sigma-1}}S_{R_\sigma\cap R_{\sigma-1}}S_{R^+_{\sigma-1}\cup R^-_\sigma} = \mathrm{E}\, S_{R_\sigma\cap R_{\sigma-1}}\cdot \mathrm{E}\, S_{R^-_\sigma}S_{R^+_{\sigma-1}}S_{R^+_{\sigma-1}\cup R^-_\sigma} = 0,$$

because either the first or the second expectation equals zero. Substituting this results in

$$\frac{1}{2}\mathrm{E}\, Y_a^2 = \frac{1}{2}\sum_{\sigma=1}^s 4^{\mathbb{1}_{\{\sigma\geq 2\}}}(t^*_\sigma)^2\,\mathrm{E}\Big[S^2_{R^-_\sigma}S^2_{R^+_{\sigma-1}} + S^2_{R_{\sigma-1}\cap R_\sigma}S^2_{R^-_\sigma\cup R^+_{\sigma-1}}\Big] + e_1(\underline{t}^*). \tag{55}$$

Furthermore, because

$$\mathrm{E}[S^2_{R^-_\sigma}S^2_{R^+_{\sigma-1}} + S^2_{R_{\sigma-1}\cap R_\sigma}S^2_{R^-_\sigma\cup R^+_{\sigma-1}}] = |R^-_\sigma||R^+_{\sigma-1}| + (|R^-_\sigma| + |R^+_{\sigma-1}|)|R_\sigma\cap R_{\sigma-1}|$$

and

$$|R^-_\sigma||R^+_{\sigma-1}| + (|R^-_\sigma| + |R^+_{\sigma-1}|)|R_\sigma\cap R_{\sigma-1}| = |R_\sigma||R_{\sigma-1}| - |R_\sigma\cap R_{\sigma-1}|^2 \leq |R_\sigma||R_{\sigma-1}|,$$

the first term in the right-hand side of (55) is bounded from above by

$$\frac{1}{2}\sum_{\sigma=1}^s 4^{\mathbb{1}_{\{\sigma\geq 2\}}}(t^*_\sigma)^2|R_\sigma||R_{\sigma-1}| \leq \frac{1}{2}\sum_{\sigma=1}^s \frac{4^{-\mathbb{1}_{\{\sigma\geq 2\}}}|R_\sigma|}{|R_{\sigma-1}|} = \mathcal{H}.$$

Concerning $e_1(\underline{t}^*)$, the following lemma will be useful.

Lemma 2. *Let $A_1, \ldots, A_4 \subset \mathbb{N} \cup \{0\}$. If $A_1 \cap A_2 = \emptyset$ and $A_3 \cap A_4 = \emptyset$, then*
(i)

$$0 \leq \mathrm{E}\, S_{A_1} S_{A_2} S_{A_3} S_{A_4} \leq |A_3|\,|A_4|,$$

(ii)

$$\mathrm{E}\, S_{A_1}^2 S_{A_3} S_{A_4} = 2|A_1 \cap A_3||A_1 \cap A_4| \leq 2 \min\{|A_1|^2, |A_3|\,|A_4|\}.$$

Proof. See Appendix A.

We observe that for every σ, σ' four cross terms are present in $e_1(\underline{t}^*)$. Each cross term is the expectation over the product of four sums, and fits precisely in Lemma 2. For example, take for the first cross term $A_1 = R_\sigma^-$, $A_2 = R_{\sigma-1}^+$, $A_3 = R_{\sigma'}^-$ and $A_4 = R_{\sigma'-1}^+$, then from Lemma 2 it follows

$$0 \leq \mathrm{E}\, S_{R_\sigma^-} S_{R_{\sigma-1}^+} S_{R_{\sigma'}^-} S_{R_{\sigma'-1}^+} \leq |R_{\sigma'}||R_{\sigma'-1}|.$$

Thus, using that $t_\sigma^* \leq 0$ for all σ and applying Lemma 2 repeatedly gives

$$|e_1(\underline{t}^*)| \leq C \sum_{1 \leq \sigma < \sigma' \leq s} t_\sigma^* t_{\sigma'}^* |R_{\sigma'}||R_{\sigma'-1}| = C \sum_{1 \leq \sigma < \sigma' \leq s} \frac{|R_{\sigma'}|}{|R_{\sigma-1}|} \leq C \sum_{1 \leq \sigma < \sigma' \leq s} \frac{|R_{\sigma'}|}{|R_{\sigma'-1}|} \frac{|R_\sigma|}{|R_{\sigma-1}|} \leq C\mathcal{H}^2,$$

since $\sigma \leq \sigma' - 1$ and thus $|R_\sigma| \geq |R_{\sigma'-1}|$. We have shown that

$$h(\underline{t}^*) \leq 1 - 2\mathcal{H} + \mathcal{H} + \mathcal{O}(\mathcal{H}^2) + e(\underline{t}^*) = 1 - \mathcal{H} + \mathcal{O}(\mathcal{H}^2) + e(\underline{t}^*).$$

To complete the proof of (i), it is sufficient to prove that $\mathrm{E}\, e^{3|Y_a|}$, $\mathrm{E}\, e^{12|Y_a|}$ and $\mathrm{E}\, e^{2Y_a}$ are bounded, $\mathrm{E}\, Y_a^6 = \mathcal{O}(\mathcal{H}^3)$, $\mathrm{E}\, Y_q Y_a = \mathcal{O}(\mathcal{H}^2)$, $\mathrm{E}\, Y_a^3 \leq 0$ and $\mathrm{E}\, Y_q^4 = \mathcal{O}(\mathcal{H}^4)$. Indeed, by (54) it then follows that $e(\underline{t}) \leq \mathcal{O}(\mathcal{H}^2)$. The remainder of the proof of (i) is focused on the five statements above. Using $\mathrm{E}\, e^{x|Y_a|} \leq \mathrm{E}\, e^{xY_a} + \mathrm{E}\, e^{-xY_a}$ and following the argument of (44), the moment generating functions $\mathrm{E}\, e^{3|Y_a|}$, $\mathrm{E}\, e^{12|Y_a|}$ and $\mathrm{E}\, e^{2Y_a}$ are uniformly bounded by Lemma 1. Furthermore, by independence of $S_{R_\sigma^-}$ and $S_{R_{\sigma-1}^+}$, and of $S_{R_{\sigma-1} \cap R_\sigma}$ and $S_{R_\sigma^- \cup R_{\sigma-1}^+}$,

$$\begin{aligned}
\mathrm{E}\, Y_a^6 &\leq C \sum_{\sigma=1}^{s} (t_\sigma^*)^6 \Big[\mathrm{E}\, S_{R_\sigma^-}^6\, \mathrm{E}\, S_{R_{\sigma-1}^+}^6 + \mathrm{E}\, S_{R_{\sigma-1} \cap R_\sigma}^6\, \mathrm{E}\, S_{R_\sigma^- \cup R_{\sigma-1}^+}^6 \Big] \\
&\leq C \sum_{\sigma=1}^{s} (t_\sigma^*)^6 \Big[|R_\sigma^-|^3 |R_{\sigma-1}^+|^3 + |R_{\sigma-1} \cap R_\sigma|^3 |R_\sigma^- \cup R_{\sigma-1}^+|^3 \Big] \leq C \sum_{\sigma=1}^{s} \frac{|R_\sigma|^3}{|R_{\sigma-1}|^3} \leq C\mathcal{H}^3.
\end{aligned}$$

Similarly, $\mathrm{E}\, Y_q^4$ is bounded by

$$C \sum_{\sigma=1}^{s} (t_\sigma^*)^4 \Big[|R_\sigma|^4 + \mathrm{E}\, S_{R_\sigma \cap R_{\sigma-1}}^8 \Big] \leq C \sum_{\sigma=1}^{s} (t_\sigma^*)^4 \Big[|R_\sigma|^4 + |R_\sigma \cap R_{\sigma-1}|^4 \Big] \leq C\mathcal{H}^4.$$

Concerning $\mathrm{E}\, Y_a^3$, we have

$$\begin{aligned}
\mathrm{E}\, Y_a^3 &= \sum_{\sigma, \sigma', \sigma''} t_\sigma^* t_{\sigma'}^* t_{\sigma''}^*\, \mathrm{E} \Big\{ \Big[S_{R_\sigma^-} S_{R_{\sigma-1}^+} + S_{R_{\sigma-1} \cap R_\sigma} S_{R_\sigma^- \cup R_{\sigma-1}^+} \Big] \\
&\quad \times \Big[S_{R_{\sigma'}^-} S_{R_{\sigma'-1}^+} + S_{R_{\sigma'-1} \cap R_{\sigma'}} S_{R_{\sigma'}^- \cup R_{\sigma'-1}^+} \Big] \Big[S_{R_{\sigma''}^-} S_{R_{\sigma''-1}^+} + S_{R_{\sigma''-1} \cap R_{\sigma''}} S_{R_{\sigma''}^- \cup R_{\sigma''-1}^+} \Big] \Big\}.
\end{aligned}$$

Since all $t_\sigma^* \leq 0$ and

$$\mathrm{E} \prod_{i=1}^{l} S_{A_i} \geq 0 \quad \text{for all } l,\ A_1, \ldots, A_l \in \mathbb{N}. \tag{56}$$

we conclude that $\mathrm{E}\, Y_a^3 \leq 0$. Finally, $\mathrm{E}\, Y_q Y_a$ equals

$$\begin{aligned}
&\mathrm{E} \Big\{ \sum_{1 \leq \sigma \leq s} t_\sigma \Big[|R_\sigma| - 2^{\mathbb{1}_{\{\sigma \geq 2\}}} |R_\sigma \cap R_{\sigma-1}| + 2^{\mathbb{1}_{\{\sigma \geq 2\}}} S_{R_\sigma \cap R_{\sigma-1}}^2 \Big] \\
&\qquad \times \sum_{1 \leq \sigma' \leq s} t_{\sigma'} 2^{\mathbb{1}_{\{\sigma' \geq 2\}}} \Big[S_{R_{\sigma'}^-} S_{R_{\sigma'-1}^+} + S_{R_{\sigma'-1} \cap R_{\sigma'}} S_{R_{\sigma'}^- \cup R_{\sigma'-1}^+} \Big] \Big\} \\
&= \sum_{1 \leq \sigma, \sigma' \leq s} t_\sigma t_{\sigma'} 2^{\mathbb{1}_{\{\sigma \geq 2\}} + \mathbb{1}_{\{\sigma' \geq 2\}}}\, \mathrm{E}\, S_{R_\sigma \cap R_{\sigma-1}}^2 \Big[S_{R_{\sigma'}^-} S_{R_{\sigma'-1}^+} + S_{R_{\sigma'-1} \cap R_{\sigma'}} S_{R_{\sigma'}^- \cup R_{\sigma'-1}^+} \Big].
\end{aligned}$$

By Lemma 2 (ii), we can bound this by

$$\sum_{1\le\sigma'\le\sigma\le s} t_\sigma t_{\sigma'} 2^{\mathbb{1}_{\{\sigma\ge 2\}}+\mathbb{1}_{\{\sigma'\ge 2\}}} 2|R_\sigma\cap R_{\sigma-1}|^2 + \sum_{1\le\sigma<\sigma'\le s} t_\sigma t_{\sigma'} 2^{\mathbb{1}_{\{\sigma\ge 2\}}+\mathbb{1}_{\{\sigma'\ge 2\}}} 2|R_{\sigma'}\cap R_{\sigma'-1}||R^+_{\sigma'-1}\cup R^-_{\sigma'}|$$

$$\le \quad C\sum_{1\le\sigma'\le\sigma\le s}\frac{|R_\sigma|^2}{|R_{\sigma-1}||R_{\sigma'-1}|} + C\sum_{1\le\sigma<\sigma'\le s}\frac{|R_{\sigma'}||R_{\sigma'-1}|}{|R_{\sigma-1}||R_{\sigma'-1}|}\le C\mathcal{H}^2,$$

since $|R_{\sigma'-1}|\ge|R_{\sigma-1}|$ whenever $\sigma'\le\sigma$ and $|R_{\sigma'}|\le|R_{\sigma+1}|$ for $\sigma'\ge\sigma+1$.
Step 2: Proof of the upper bound (ii). An essential ingredient in the proof of the upper bound for $s=2$ was exchangeability. For $s>2$, the set of possible configurations $\underline{R}$ is large and this prevents us from using exchangeability. Therefore we restrict ourselves to $R_\sigma=\{\sum_{\sigma'=\sigma+1}^s|R_{\sigma'}|+1,\dots,\sum_{\sigma'=\sigma+1}^s|R_{\sigma'}|\}$, i.e., we take the R_σ's to be disjoint. By exchangeability, we have that all users within one of the groups R_σ or the group $R_0^*=R_0\backslash\{\cup_{\sigma=1}^s R_\sigma\}$ behave the same. We first write (53) as

$$-\lim_{n\to\infty}\frac{1}{n}\log\mathrm{P}\Big(\bigcap_{1\le\sigma\le s}\Big\{\max_{m\in R_\sigma}\bar{Z}_m^{(\sigma)}\le 0\Big\}\bigcap_{\substack{1\le\sigma\le s-1\\1\le\sigma'\le s,\,\sigma'\ne\sigma}}\Big\{\min_{m\in R_{\sigma'}}\bar{Z}_m^{(\sigma)}\ge 0\Big\}\bigcap_{1\le\sigma\le s-1}\Big\{\min_{m\in R_0^*}\bar{Z}_m^{(\sigma)}\ge 0\Big\}\Big).$$

Existence of $H^{(s)}_{k,\underline{R}}$ follows from Cramér's Theorem. Using convexity of a suitably chosen I_r (as in (26)), we can sum over the users in each group, so that we can prove that $H^{(s)}_{k,\underline{R}}$ equals

$$-\lim_{n\to\infty}\frac{1}{n}\log\mathrm{P}\Big(\bigcap_{1\le\sigma\le s}\Big\{\sum_{m\in R_\sigma}\bar{Z}_m^{(\sigma)}\le 0\Big\}\bigcap_{\substack{1\le\sigma\le s-1\\1\le\sigma'\le s,\,\sigma'\ne\sigma}}\Big\{\sum_{m\in R_{\sigma'}}\bar{Z}_m^{(\sigma)}\ge 0\Big\}\bigcap_{1\le\sigma\le s-1}\Big\{\sum_{m\in R_0^*}\bar{Z}_m^{(\sigma)}\ge 0\Big\}\Big).$$

The events $\{\sum_{m\in R_\sigma}\bar{Z}_m^{(\sigma)}\ge 0\}$, $1\le\sigma'<\sigma\le s-1$ and $\{\sum_{m\in R_0^*}\bar{Z}_m^{(\sigma)}\ge 0\}$, $1\le\sigma\le s-1$ turn out not to contribute to the rate, even though we can only prove this when $\mathcal{H}$ is sufficiently small, as shown in Appendix B. Here we suffice with the statement of the result.

Lemma 3. *Let*

$$\tilde{H}^{(s)}_{k,\underline{R}}=-\lim_{n\to\infty}\frac{1}{n}\log\mathrm{P}\left(\bigcap_{1\le\sigma\le s}\Big\{\sum_{m\in R_\sigma}\bar{Z}_m^{(\sigma)}\le 0\Big\}\bigcap_{1\le\sigma<\sigma'\le s}\Big\{\sum_{m\in R_{\sigma'}}\bar{Z}_m^{(\sigma)}\ge 0\Big\}\right).$$

Then, for k sufficiently large,

$$H^{(s)}_{k,\underline{R}}=\tilde{H}^{(s)}_{k,\underline{R}}.$$

We will prove that at stage σ, only the block R_σ contributes to the first order of the rate. The blocks $R_{\sigma'}$, $\sigma'>\sigma$ contribute only in lower order. In Figure 4 the situation is shown. We are left with an $s(s+1)/2$ dimensional problem that reads

$$H^{(s)}_{k,\underline{R}}\le-\lim_{n\to\infty}\frac{1}{n}\log\mathrm{P}\Big(\bigcap_{1\le\sigma\le s}\Big\{\frac{1}{n}\sum_{i=1}^n Y_{(\sigma,\sigma),i}\le 0\Big\}\bigcap_{1\le\sigma<\sigma'\le s}\Big\{\frac{1}{n}\sum_{i=1}^n Y_{(\sigma,\sigma'),i}\ge 0\Big\}\Big),$$

where $(Y_i)_{i=1}^n$ are i.i.d. and $s(s+1)/2$ dimensional and its components are given by

$$Y_{(\sigma,\sigma'),i}=\begin{cases}\displaystyle\sum_{m\in R_0}X_{mi}\sum_{m\in R_{\sigma'}}X_{mi}, & \sigma=1,\ 1\le\sigma'\le s,\\ \displaystyle|R_{\sigma'}|+2\sum_{m\in R_{\sigma-1}}X_{mi}\sum_{m\in R_{\sigma'}}X_{mi}, & 2\le\sigma\le\sigma'\le s.\end{cases}$$

Let $h(\underline{t})=\mathrm{E}\,e^{\langle\underline{t},Y_1\rangle}$. Cramér's Theorem gives

$$H^{(s)}_{k,\underline{R}}\le\sup_{\underline{t}\in D}\{-\log h(\underline{t})\},$$

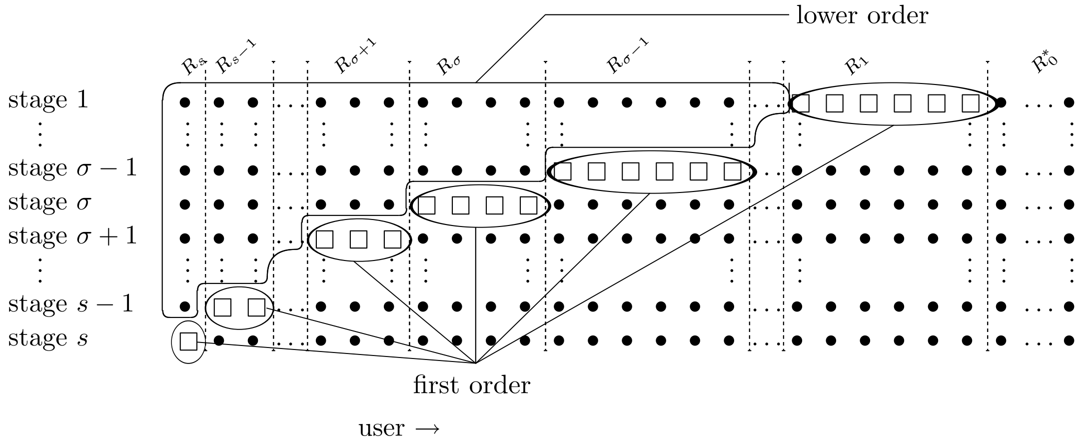

FIGURE 4: Configuration, $\square$ means $\bar{Z}_m^{(\sigma)} \le 0$, $\bullet$ means $\bar{Z}_m^{(\sigma)} \ge 0$.

where

$$D = \{\underline{t} : t_{\sigma,\sigma} \le 0 \text{ for all } 1 \le \sigma \le s, \text{ while } t_{\sigma,\sigma'} \ge 0 \text{ for all } 1 \le \sigma < \sigma' \le s\}. \tag{57}$$

The domain D arises from the form of the event of interest $\frac{1}{n}\sum Y_{i,(\sigma,\sigma)} \le 0$, $\frac{1}{n}\sum Y_{i,(\sigma,\sigma')} \ge 0$ for all $\sigma' \ne \sigma$. To prove the claim in (ii), we will define an appropriate ellipse $\mathcal{E}$, with $\underline{0} \in \mathcal{E}^0$, the interior of $\mathcal{E}$. Similarly to the proof of the upper bound on $H_{k,r}^{(2)}$, we will show that $h(\underline{t}) > 1$ for all $\underline{t} \in \partial\mathcal{E} \cap D$, which implies that the infimum over $h(\underline{t})$ is attained in $\mathcal{E}^0 \cap D$ since $h(\underline{0}) = 1$. Indeed, whenever $\underline{t} \in D$, but $\underline{t} \notin \mathcal{E}$, there exists a unique $0 < \alpha < 1$ such that $\alpha\underline{t} \in \partial\mathcal{E} \cap D$. Convexity of h and $h(\alpha\underline{t}) > 1$ implies

$$1 < h(\alpha\underline{t}) = h(\alpha\underline{t} + (1-\alpha)\underline{0}) \le \alpha h(\underline{t}) + (1-\alpha)h(\underline{0}) = \alpha h(\underline{t}) + (1-\alpha)$$

so that indeed $h(\underline{t}) > 1$.

We write $\langle \underline{t}, \underline{Y}_1 \rangle = Y_q + Y_a$, where

$$Y_q = \sum_{\sigma'=1}^{s} t_{1,\sigma'} S_{R_{\sigma'}}^2 + \sum_{2\le\sigma\le\sigma'\le s} t_{\sigma,\sigma'} |R_{\sigma'}|, \tag{58}$$

$$\begin{aligned} Y_a &= \sum_{\sigma=1}^{s} t_{1\sigma} S_{R_0\setminus R_\sigma} S_{R_\sigma} + 2 \sum_{2\le\sigma\le\sigma'\le s} t_{\sigma,\sigma'} S_{R_{\sigma-1}} S_{R_{\sigma'}} \\ &= \Big(t_{11} S_{R_0\setminus R_1} S_{R_1} + 2\sum_{\sigma=2}^{s} t_{\sigma,\sigma} S_{R_{\sigma-1}} S_{R_\sigma} \Big) + \Big(\sum_{\sigma=2}^{s} t_{1\sigma} S_{R_\sigma} S_{R_0\setminus R_\sigma} + 2 \sum_{2\le\sigma<\sigma'\le s} t_{\sigma,\sigma'} S_{R_{\sigma-1}} S_{R_{\sigma'}} \Big), \end{aligned} \tag{59}$$

where we have split according to the signs of the t's. We have

$$h(\underline{t}) = 1 + \mathrm{E}\, Y_q + \frac{1}{2}\mathrm{E}\, Y_a^2 + e(\underline{t}),$$

where, using (40) and $\mathrm{E}\, Y_a = 0$,

$$e(\underline{t}) \ge \mathrm{E}\Big[Y_a^3/6 + Y_q(Y_a + Y_a^2/2 + Y_a^3/6 + Y_a^4 e^{\zeta Y_a}/24) \Big].$$

Observe that for all $2 \le \sigma \le \sigma' < \sigma'' \le s$,

$$\begin{aligned} \mathrm{E}\, S_{R_0\setminus R_1} S_{R_1} S_{R_0\setminus R_\sigma} S_{R_\sigma} &= |R_1||R_\sigma|, \\ \mathrm{E}\, S_{R_0\setminus R_1} S_{R_1} S_{R_{\sigma-1}} S_{R_{\sigma'}} &= \begin{cases} |R_1||R_{\sigma'}| & \sigma = 2 \\ 0 & \text{elsewhere} \end{cases} \\ \mathrm{E}\, S_{R_{\sigma-1}} S_{R_\sigma} S_{R_0\setminus R_{\sigma'}} S_{R_{\sigma'}} &= \begin{cases} |R_{\sigma-1}||R_\sigma| & \sigma' = \sigma-1 \text{ or } \sigma' = \sigma \\ 0 & \text{elsewhere} \end{cases} \\ \mathrm{E}\, S_{R_{\sigma-1}} S_{R_\sigma} S_{R_{\sigma'-1}} S_{R_{\sigma''}} &= 0. \end{aligned}$$

Together with (56) and $t_{\sigma,\sigma} \leq 0$, while $t_{\sigma,\sigma'} \geq 0$ for all $\sigma < \sigma'$, we arrive at

$$\begin{aligned}
\mathrm{E}\, Y_q &= \sum_{\sigma=1}^{s}\sum_{\sigma'=\sigma}^{s} t_{\sigma,\sigma'}|R_{\sigma'}| \\
\mathrm{E}\, Y_a^2 &\geq t_{11}^2|R_0\backslash R_1||R_1| + 4\sum_{\sigma=2}^{s} t_{\sigma,\sigma}^2|R_{\sigma-1}||R_\sigma| \qquad (60)\\
&\quad + \sum_{\sigma'=2}^{s} t_{1,\sigma'}^2|R_0\backslash R_{\sigma'}|\,|R_{\sigma'}| + 4\sum_{2\leq\sigma<\sigma'\leq s} t_{\sigma,\sigma'}^2|R_{\sigma-1}||R_{\sigma'}| \\
&\quad + \sum_{\sigma'=2}^{s} t_{1,\sigma'}\Big(t_{11}|R_1||R_{\sigma'}| + 2t_{\sigma',\sigma'}|R_{\sigma'}||R_{\sigma'-1}| + 2t_{\sigma'+1,\sigma'+1}|R_{\sigma'}||R_{\sigma'+1}|\Big) \\
&\quad + \sum_{\sigma'=3}^{s} t_{2,\sigma'}t_{11}|R_1||R_{\sigma'}|.
\end{aligned}$$

The next goal is to write $h(\underline{t})$ as a sum of squares, similarly to (46). However, since the situation is more involved, we need some more abbreviations. First of all, we introduce $t^*_{\sigma,\sigma'}$ as follows:

$$t^*_{\sigma,\sigma'} = \begin{cases} -\frac{1}{|R_0\backslash R_{\sigma'}|}, & \sigma=\sigma'=1, \\ -\frac{1}{4|R_{\sigma-1}|}, & 2\leq\sigma=\sigma'\leq s, \\ 0 & 2\leq\sigma<\sigma'\leq s. \end{cases}$$

Below we will prove that these t^*'s are asymptotically close to the minimizers of the minimization problem $\sup_{\underline{t}\in D}\{-\log h(\underline{t})\}$. We remark that Figure 4 then indeed depicts the correct situation, as $t^*_{\sigma,\sigma'} \neq 0$ only when $\sigma=\sigma'$. Furthermore, we introduce a slight variation on $\mathcal{H}$ (recall (51)):

$$\mathcal{H}' = \frac{1}{2}\frac{|R_1|}{|R_0\backslash R_1|} + \frac{1}{8}\sum_{\sigma=2}^{s}\frac{|R_\sigma|}{|R_{\sigma-1}|}.$$

Since $1/|R_0\backslash R_1| = (1+\mathcal{O}(|R_1|/|R_0|))/|R_0|$, we have that $\mathcal{H}' = \mathcal{H}(1+\mathcal{O}(\mathcal{H}))$. After completing the squares (where we have incorporated all cross terms $t_{1,\sigma}t_{\sigma,\sigma}$) we arrive at

$$h(\underline{t}) = 1 + \mathrm{E}\, Y_q + \mathrm{E}\, Y_a^2/2 + e(\underline{t}) \geq 1 - \mathcal{H}' + Q_{\underline{R}}(\underline{t}) + e_1(\underline{t}), \qquad (61)$$

where

$$\begin{aligned}
Q_{\underline{R}}(\underline{t}) &= \frac{1}{2}|R_0\backslash R_1||R_1|(t_{11}-t^*_{11})^2 + 2\sum_{\sigma=2}^{s}|R_{\sigma-1}||R_\sigma|\Big(1-\frac{1}{2}\frac{|R_{\sigma-1}|}{|R_0\backslash R_\sigma|}\Big)(t_{\sigma,\sigma}-t^*_{\sigma,\sigma})^2 \\
&\quad + \frac{1}{8}\sum_{\sigma'=2}^{s}|R_0\backslash R_{\sigma'}|\,|R_{\sigma'}|(t_{1,\sigma'}-t^*_{1,\sigma'})^2 + \sum_{2\leq\sigma<\sigma'\leq s}|R_{\sigma-1}||R_{\sigma'}|(t_{\sigma,\sigma'}-t^*_{\sigma,\sigma'})^2.
\end{aligned}$$

and

$$\begin{aligned}
e_1(\underline{t}) &\geq \frac{3}{4}\sum_{\sigma'=2}^{s} t_{1,\sigma'}|R_{\sigma'}| + \sum_{2\leq\sigma<\sigma'\leq s} t_{\sigma,\sigma'}|R_{\sigma'}| + \frac{1}{8}\sum_{\sigma'=2}^{s}|R_0\backslash R_{\sigma'}||R_{\sigma'}|t^2_{1,\sigma'} \\
&\quad + \frac{1}{4}\sum_{\sigma'=2}^{s}|R_0\backslash R_{\sigma'}||R_{\sigma'}|\Big(t_{1,\sigma'} + 2\frac{|R_{\sigma'-1}|}{|R_0\backslash R_{\sigma'}|}(t_{\sigma',\sigma'}-t^*_{\sigma',\sigma'})\Big)^2 + \sum_{2\leq\sigma<\sigma'\leq s}|R_{\sigma-1}||R_{\sigma'}|t^2_{\sigma,\sigma'} \\
&\quad + \frac{1}{2}\sum_{\sigma'=2}^{s} t_{1,\sigma'}\Big(t_{11}|R_1||R_{\sigma'}| + 2t_{\sigma'+1,\sigma'+1}|R_{\sigma'}||R_{\sigma'+1}|\Big) + \frac{1}{2}\sum_{\sigma'=3}^{s} t_{2,\sigma'}t_{11}|R_1||R_{\sigma'}| \\
&\quad + \mathrm{E}\, Y_a^3/6 + \mathrm{E}\, Y_q(Y_a + Y_a^2/2 + Y_a^3/6 + Y_a^4 e^{\zeta Y_a}/24).
\end{aligned}$$

This is bounded from below by

$$\begin{aligned}
& e_2(\underline{t}) && (62)\\
&= \frac{1}{8}\sum_{\sigma'=2}^{s} |R_0\backslash R_{\sigma'}||R_{\sigma'}|t_{1,\sigma'}^2 + \sum_{2\le\sigma<\sigma'\le s} |R_{\sigma-1}||R_{\sigma'}|t_{\sigma,\sigma'}^2 && (63)\\
&\quad + \sum_{\sigma'=2}^{s} t_{1,\sigma'}\left(\frac{3}{4}|R_{\sigma'}| + \frac{1}{2}t_{11}|R_1||R_{\sigma'}| + t_{\sigma'+1,\sigma'+1}|R_{\sigma'}||R_{\sigma'+1}|\right)\\
&\quad + \sum_{\sigma'=3}^{s} t_{2,\sigma'}\left(|R_{\sigma'}| + \frac{1}{2}t_{11}|R_1||R_{\sigma'}|\right) + \mathrm{E}\, Y_a^3/6 + \mathrm{E}\, Y_q(Y_a + Y_a^2/2 + Y_a^3/6 + Y_a^4 e^{\zeta Y_a}/24).
\end{aligned}$$

We next define the ellipse to be

$$\mathcal{E} \quad = \quad \left\{\underline{t} : Q_{\underline{R}}(\underline{t}) \le 2\mathcal{H}'\right\}. \tag{64}$$

We can derive from (64) that for all $\underline{t} \in \mathcal{E}$,

$$|t_{1,\sigma}| \quad \le \quad 3\mathcal{H}'^{1/2}\left[\frac{1}{|R_0\backslash R_\sigma||R_\sigma|}\right]^{1/2},\ 1 \le \sigma \le s \tag{65}$$

$$|t_{\sigma,\sigma'}| \quad \le \quad 3\mathcal{H}'^{1/2}\left[\frac{1}{|R_{\sigma'}||R_{\sigma-1}|}\right]^{1/2},\ 2 \le \sigma \le \sigma' \le s. \tag{66}$$

Now we are in the position to bound $e_2(\underline{t})$ for $\underline{t} \in \mathcal{E}$. However, this is quite involved. For this reason, we state the result in the next lemma, while we transfer the proof to Appendix C.

Lemma 4. *There exists a C, depending neither on k nor on $\underline{R}$, such that for $\underline{t} \in \mathcal{E}$,*

$$e_2(\underline{t}) \ge -C\mathcal{H}^2.$$

When k is sufficiently large (and thus $\mathcal{H}'$ sufficiently small), we now have that (recall (61) and (64))

$$h(\underline{t}) \ge 1 + \mathcal{H}'(-1 + 2 - C\mathcal{H}') > 1 \quad \text{for} \quad \underline{t} \in \partial\mathcal{E} \cap D$$

and we needed this to prove that the supremum is attained in $\mathcal{E}^0 \cap D$. We have, according to (61), for all $\underline{t} \in \mathcal{E} \cap D$,

$$h(\underline{t}) \quad \ge \quad 1 - \mathcal{H}' + Q_{\underline{R}}(\underline{t}) - C\mathcal{H}'^2.$$

It is clear that the infimum of the right-hand side of the equation above is attained at $\underline{t} = \underline{t}^*$, where $Q_{\underline{R}}(\underline{t}) = 0$, so that

$$H_{k,\underline{R}}^{(s)} \le -\log(1 - \mathcal{H}' + \mathcal{O}(\mathcal{H}'^2)) = \mathcal{H}'(1 + \mathcal{O}(\mathcal{H}')).$$

Since $\mathcal{H}' = \mathcal{H}(1 + \mathcal{O}(\mathcal{H}))$, this completes the proof.

6. Proof of Corollary 2

The proof is analogous to the proof of part (i) of Proposition 2 and is therefore only briefly sketched. First of all, we are allowed to replace max by $\sum$ (because $\mathrm{P}(\max \cdot \le 0) \le \mathrm{P}(\sum \cdot \le 0)$). Thus, we are interested in

$$-\lim_{n\to\infty} \frac{1}{n} \log \mathrm{P}\left(\frac{1}{n}\sum_{i=1}^{n} Y_i \le 0\right) = \max_{t\le 0}\left\{-\log \mathrm{E}\, e^{tY_1}\right\},$$

where

$$Y_i = \alpha|A_1| - |A_1 \cap A_2| + \sum_{m\in A_1} X_{mi} \sum_{j\in A_2} X_{ji}.$$

We write $tY_1 = Y_q + Y_a$, where $Y_q \stackrel{d}{=} t(\alpha|A_1| - |A_1 \cap A_2| + S_{A_1 \cap A_2})$ represents the quadratic part and $Y_a \stackrel{d}{=} t(S_{A_1 \setminus A_2} S_{A_2 \setminus A_1} + S_{A_1 \setminus A_2 \cup A_2 \setminus A_1} S_{A_1 \cap A_2})$ represents the asymmetric part with mean zero. We substitute $t^* = -\alpha/|A_2|$, which will lead to a lower bound of the rate. We write (recall (39))

$$h(t^*) = \mathrm{E}\, e^{t^* Y_1} = 1 + \mathrm{E}\, Y_q + \frac{1}{2}\, \mathrm{E}\, Y_a^2 + e(t^*),$$

where $e(t^*)$ is bounded by (42). Using the same techniques as in part (i) of Proposition 2, we can show that $\mathrm{E}\, Y_q + \frac{1}{2}\, \mathrm{E}\, Y_a^2 = \alpha^2 |A_1|/(2|A_2|) + \mathcal{O}(|A_1|^2/|A_2|^2)$ and that $e(t^*) = \mathcal{O}(|A_1|^2/|A_2|^2)$.

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Appendix A. Proof of Lemma 2

Proof of Lemma 2 (i). We will prove that $\mathrm{E}\, S_{A_1} S_{A_2} S_{A_3} S_{A_4} \leq |A_3||A_4|$. Note that $\mathrm{E}\, S_{A_1} \leq 1$ and that $\mathrm{E}\, S_{A_1} S_{A_3} = |A_1 \cap A_3|$, regardless whether $\{0\} \in A_1 \cup A_3$ or not. By symmetry in A_3 and A_4, we may assume that $0 \notin A_4$. We perform induction in $|A_4|$. When $A_4 = \{i\} \neq \{0\}$, we have

$$\mathrm{E}\, S_{A_1} S_{A_2} S_{A_3} U_i = \begin{cases} 0 & \{i\} \notin A_1 \cup A_2 \cup A_3, \\ \mathrm{E}\, S_{A_1}\, \mathrm{E}\, S_{A_2} S_{A_3} \leq |A_2 \cap A_3| \leq |A_3| & \{i\} \in A_1, \{i\} \notin A_2 \cup A_3, \\ \mathrm{E}\, S_{A_2}\, \mathrm{E}\, S_{A_1} S_{A_3} \leq |A_1 \cap A_3| \leq |A_3| & \{i\} \in A_2, \{i\} \notin A_1 \cup A_3, \end{cases}$$

so that for $|A_4| = 1$ the claim is true. Next, we write $A_4 = A_4' \cup \{i\}$ for some $i \neq 0$. By construction, i cannot be in A_3, but it can be in A_1, A_2 but not both. Suppose the claim is true for $|A_4| = n-1$. Then for $|A_4| = n$, we have three cases.
Case 1: $\{i\} \notin A_1 \cup A_2 \cup A_3$. Then

$$\mathrm{E}\, S_{A_1} S_{A_2} S_{A_3} S_{A_4} = \mathrm{E}\, S_{A_1} S_{A_2} S_{A_3} S_{A_4'} \leq |A_3||A_4'| \leq |A_3||A_4|.$$

Case 2: $\{i\} \in A_1$, $\{i\} \notin A_2 \cup A_3$. Then $S_{A_1} = S_{A_1'} + U_i$ and $S_{A_4} = S_{A_4'} + U_i$, so that

$$\mathrm{E}\, S_{A_1} S_{A_2} S_{A_3} S_{A_4} = \mathrm{E}\, S_{A_2} S_{A_3} + \mathrm{E}\, S_{A_1'} S_{A_2} S_{A_3} S_{A_4'} \leq |A_2 \cap A_3| + |A_3||A_4'| \leq |A_3||A_4|.$$

Case 3: $\{i\} \in A_2$, $\{i\} \notin A_1 \cup A_3$. This is identical to case 2, where A_1 and A_2 are interchanged. We conclude by induction that the claim holds for all A_4.

Proof of Lemma 2 (ii). Compute

$$\begin{aligned} &\mathrm{E}\, S_{A_1}^2 S_{A_3} S_{A_4} \\ &= \mathrm{E}\, S_{A_1}^2 S_{A_3\setminus A_1} S_{A_4\setminus A_1} + \mathrm{E}\, S_{A_1}^2 S_{A_3\cap A_1} S_{A_4\setminus A_1} + \mathrm{E}\, S_{A_1}^2 S_{A_3\setminus A_1} S_{A_4\cap A_1} + \mathrm{E}\, S_{A_1}^2 S_{A_3\cap A_1} S_{A_4\cap A_1} \\ &= \mathrm{E}\, S_{A_1}^2 S_{A_3\cap A_1} S_{A_4\cap A_1} \\ &= 2\,\mathrm{E}\, S_{A_1\cap A_3} S_{A_1\cap A_4} S_{A_3\cap A_1} S_{A_4\cap A_1} = 2|A_1\cap A_3||A_1\cap A_4|. \end{aligned}$$

Appendix B. Proof of Lemma 3

For convenience, we introduce the event

$$B_n = \bigcap_{1\leq\sigma\leq s} \Big\{ \sum_{m\in R_\sigma} \bar{Z}_{m+1}^{(\sigma)} \leq 0 \Big\} \bigcap_{1\leq\sigma<\sigma'\leq s} \Big\{ \sum_{m\in R_{\sigma'}} \bar{Z}_{m+1}^{(\sigma)} \geq 0 \Big\}$$

By definition, $\tilde{H}_{k,\underline{R}}^{(s)} = -\lim_{n\to\infty} \frac{1}{n} \log \mathrm{P}(B_n)$. Clearly,

$$\begin{aligned} \mathrm{P}(B_n) &= \mathrm{P}\Big(B_n \bigcap_{1\leq\sigma'<\sigma\leq s-1} \Big\{ \sum_{m\in R_{\sigma'}} \bar{Z}_{m+1}^{(\sigma)} \geq 0 \Big\} \bigcap_{1\leq\sigma\leq s-1} \Big\{ \sum_{m\in R_0^+} \bar{Z}_{m+1}^{(\sigma)} \geq 0 \Big\} \Big) \\ &\quad + \mathrm{P}\Big(B_n \cap \Big\{ \bigcup_{1\leq\sigma'<\sigma\leq s-1} \Big\{ \sum_{m\in R_{\sigma'}} \bar{Z}_{m+1}^{(\sigma)} \leq 0 \Big\} \bigcup_{1\leq\sigma\leq s-1} \Big\{ \sum_{m\in R_0^+} \bar{Z}_{m+1}^{(\sigma)} \leq 0 \Big\} \Big\} \Big). \end{aligned} \tag{67}$$

According to the "largest-exponent-wins" principle we have that $\tilde{H}_{k,\underline{R}}^{(s)}$ is bounded from above by the minimum of $H_{k,\underline{R}}^{(s)}$ (the rate of the first term on the right-hand side) and the rate of second term on the right-hand side. The proof is complete when we can prove that $H_{k,\underline{R}}^{(s)} \to 0$ and that the second rate is bounded from below by some $\delta > 0$ independently of k. The first fact follows from Proposition 2, even though we omitted the tilde. Concerning the rate of the second term on the right-hand side of (67), we have

$$\begin{aligned} &\mathrm{P}\Big(B_n \bigcup_{1\leq\sigma'<\sigma\leq s-1} \Big\{ \sum_{m\in R_{\sigma'}} \bar{Z}_{m+1}^{(\sigma)} \leq 0 \Big\} \bigcup_{1\leq\sigma\leq s-1} \Big\{ \sum_{m\in R_0^+} \bar{Z}_{m+1}^{(\sigma)} \leq 0 \Big\} \Big) \\ &\leq \sum_{1\leq\sigma'<\sigma\leq s-1} \mathrm{P}\Big(\sum_{m\in R_{\sigma'}} \bar{Z}_{m+1}^{(\sigma)} \leq 0 \Big) + \sum_{1\leq\sigma\leq s-1} \mathrm{P}\Big(\sum_{m\in R_0^+} \bar{Z}_{m+1}^{(\sigma)} \leq 0 \Big). \end{aligned}$$

Again we can apply the 'largest exponent wins' principle, so that it is sufficient to show that *all* rates corresponding to the probabilities on the right-hand side are larger than some $\delta > 0$. We distinguish four cases:

$$(i)\ 1 \le \sigma' = \sigma - 1 \le s-2: \quad \sum_{m \in R_{\sigma-1}} \bar{Z}^{(\sigma)}_{m+1} = \frac{1}{n}\sum_{i=1}^{n} 2\Big(\sum_{j \in R_{\sigma-1}} X_{ji}\Big)^2 - |R_{\sigma-1}|$$

$$(ii)\ 1 \le \sigma' < \sigma - 1 \le s-2: \quad \sum_{m \in R_{\sigma'}} \bar{Z}^{(\sigma)}_{m+1} = \frac{1}{n}\sum_{i=1}^{n} 2\Big(\sum_{j \in R_{\sigma'}} X_{ji}\Big)\Big(\sum_{j \in R_{\sigma-1}} X_{ji}\Big) + |R_{\sigma'}|$$

$$(iii)\ \sigma = 1: \quad \sum_{m \in R_0^+} \bar{Z}^{(1)}_{m+1} = \frac{1}{n}\sum_{i=1}^{n} \Big(\sum_{j \in R_0^+} X_{ji}\Big)^2 + \Big(\sum_{j \in R_0 \setminus R_0^+} X_{ji}\Big)\Big(\sum_{j \in R_0^+} X_{ji}\Big)$$

$$(iv)\ 2 \le \sigma \le s-1: \quad \sum_{m \in R_0^+} \bar{Z}^{(\sigma)}_{m+1} = \frac{1}{n}\sum_{i=1}^{n} 2\Big(\sum_{j \in R_0^+} X_{ji}\Big)\Big(\sum_{j \in R_{\sigma-1}} X_{ji}\Big) + |R_0^+|.$$

Note that case (ii) and (iv) are essentially the same by replacing $R_{\sigma'}$ with R_0^+, so we have to deal with three cases only. For finite $|R_{\sigma-1}|, |R_{\sigma'}|$ and $|R_0^+|$, it is clear that all rates are strictly positive. Thus, we only consider the situations where $|R_{\sigma-1}|, |R_{\sigma'}|$ and $|R_0^+|$ tend to ∞.
Case (i):

$$\begin{aligned}
&-\lim_{n\to\infty} \frac{1}{n} \log \mathrm{P}\Big(\sum_{m \in R_{\sigma-1}} \bar{Z}^{(\sigma)}_{m+1} \le 0\Big) \\
&= \sup_{t \le 0}[-\log \mathrm{E}\, e^{2tS^2_{R_{\sigma-1}} - t|R_{\sigma-1}|}] \ge -\log \mathrm{E}\, e^{2tS^2_{R_{\sigma-1}} - t|R_{\sigma-1}|}\Big|_{t=-1/(4|R_{\sigma-1}|)} \\
&= -\frac{1}{4} - \log \mathrm{E}\, e^{\frac{-1}{2|R_{\sigma-1}|} S^2_{R_{\sigma-1}}}
\end{aligned}$$

Define $Y_{A_1} = \exp(-\frac{1}{2|A_1|} S^2_{A_1})$. Then clearly $Y_{A_1} \xrightarrow{\mathcal{D}} e^{-Z^2/2}$, where $Z \stackrel{d}{=} \mathcal{N}(0,1)$ for $|A_1| \to \infty$. Furthermore, $\mathrm{E}\, Y^2_{A_1} < \infty$ for all A_1, which easily follows since $Y_{A_1} \le 1$ a.s. and $-1 < 1 - \varepsilon$. It follows from [9], Example 7.10.15 that as $|A_1| \to \infty$,

$$\mathrm{E}\, Y_{A_1} \to \mathrm{E}\, e^{-Z^2/2} = \frac{1}{\sqrt{1-2t}}\Big|_{t=-1/2} = \frac{1}{\sqrt{2}}.$$

This directly gives

$$-\lim_{n\to\infty} \frac{1}{n} \log \mathrm{P}\Big(\sum_{m \in R_{\sigma-1}} \bar{Z}^{(\sigma)}_{m+1} \le 0\Big) \ge \frac{1}{2}\log 2 - \frac{1}{4} = 0.0966 > 0.$$

Case (ii) and (iv):

$$\begin{aligned}
&-\lim_{n\to\infty} \frac{1}{n} \log \mathrm{P}\Big(\sum_{m \in R_{\sigma'}} \bar{Z}^{(\sigma)}_{m+1} \le 0\Big) = \sup_{t \le 0}[-\log \mathrm{E}\, e^{2tS_{R_{\sigma'}} S_{R_{\sigma-1}} + t|R_{\sigma'}|}] \\
&\ge \sup_{t \le 0}[-\log \mathrm{E}\, e^{2t^2 S^2_{R_{\sigma'}} |R_{\sigma-1}| + t|R_{\sigma'}|}] \ge -\log \mathrm{E}\, e^{2t^2 S^2_{R_{\sigma'}} |R_{\sigma-1}| + t|R_{\sigma'}|}\Big|_{t=-1/(4\sqrt{|R_{\sigma-1}||R_{\sigma'}|})} \\
&= \frac{|R_{\sigma'}|}{4|R_{\sigma-1}|} - \log \mathrm{E}\, e^{\frac{1}{8|R_{\sigma'}|} S^2_{R_{\sigma'}}}.
\end{aligned}$$

We bound $|R_{\sigma'}|/|R_{\sigma-1}|$ by 1, so that a similar derivation as in case (i) results in a lower bound for this rate of $\frac{1}{4} + \frac{1}{2}\log(1 - 1/4) = 0.1062 > 0$, provided that $\mathrm{E}\, e^{\frac{1}{4|R_{\sigma'}|} S^2_{R_{\sigma'}}} < \infty$. This easily follows from (45), together with $\varepsilon = 1/2$.

Case (iii):

$$\begin{aligned}
-\lim_{n\to\infty}\frac{1}{n}\log \mathrm{P}\Big(\sum_{m\in R_0^+}\bar{Z}^{(1)}_{m+1}\le 0\Big) &= \sup_{t\le 0}[-\log \mathrm{E}\, e^{tS^2_{R_0^+}+S_{R_0^+}S_{R_0\setminus R_0^+}}]\\
\ge\ \sup_{t\le 0}[-\log \mathrm{E}\, e^{tS^2_{R_0^+}+t^2S^2_{R_0^+}|R_0\setminus R_0^+|/2}] &\ge -\log \mathrm{E}\, e^{tS^2_{R_0^+}+t^2S^2_{R_0^+}|R_0\setminus R_0^+|/2}\Big|_{t=-1/(4|R_0^+|)}\\
=\ -\log \mathrm{E}\, e^{\frac{-1}{4|R_0^+|}\Big(1+\frac{|R_0\setminus R_0^+|}{2|R_0^+|}\Big)S^2_{R_0^+}}.
\end{aligned}$$

We bound $|R_0\setminus R_0^+|/|R_0^+|$ by 1, so that a similar derivation as in case (i) results in a lower bound for this rate of $\frac{1}{2}\log(1+3/8)=0.1592>0$. We conclude that if k is sufficiently large we have $\tilde{H}^{(s)}_{k,\underline{R}}=H^{(s)}_{k,\underline{R}}$.

Appendix C. Proof of Lemma 4

Recall the bounds on $t_{1,\sigma}$ and $t_{\sigma,\sigma'}$ in (65) and (66). We rearrange the terms of (62) to get

$$\begin{aligned}
e_2(\underline{t}) \;=\; &\sum_{\sigma'=2}^{s} t_{1,\sigma'}\Big(\frac{3}{4}|R_{\sigma'}|+\frac{1}{2}t_{11}|R_1||R_{\sigma'}|+t_{\sigma'+1,\sigma'+1}|R_{\sigma'}||R_{\sigma'+1}|\Big) \qquad (68)\\
&+\sum_{\sigma'=3}^{s} t_{2,\sigma'}\Big(|R_{\sigma'}|+\frac{1}{2}t_{11}|R_1||R_{\sigma'}|\Big)+\mathrm{E}\,Y_qY_a\\
&+\frac{1}{8}\sum_{\sigma'=2}^{s}|R_0\setminus R_{\sigma'}||R_{\sigma'}|t^2_{1,\sigma'}+\sum_{2\le\sigma<\sigma'\le s}|R_{\sigma-1}||R_{\sigma'}|t^2_{\sigma,\sigma'}+\mathrm{E}\,Y_a^3/6\\
&+\mathrm{E}\,Y_q(Y_a^2/2+Y_a^3/6+Y_a^4e^{\zeta Y_a}/24).
\end{aligned}$$

We will treat the terms on the different lines of (68) separately.
Step a: First line. This term is easy to bound. Indeed, for $\underline{t}\in\mathcal{E}\cap D$ (recall (57)), it follows from (65) and (66) that

$$\begin{aligned}
&t_{1,\sigma'}|R_{\sigma'}|\Big(\frac{3}{4}+\frac{1}{2}t_{11}|R_1|+t_{\sigma'+1,\sigma'+1}|R_{\sigma'+1}|\Big)\\
&\ge\ t_{1,\sigma'}|R_{\sigma'}|\Big(\frac{3}{4}-\frac{3}{2}\mathcal{H}'^{1/2}\frac{|R_1|^{1/2}}{|R_0\setminus R_1|^{1/2}}-3\mathcal{H}'^{1/2}\frac{|R_{\sigma'+1}|}{|R_0\setminus R_{\sigma'+1}|^{1/2}|R_1|^{1/2}}\Big)\\
&\ge\ t_{1,\sigma'}|R_{\sigma'}|\Big(\frac{3}{4}-C\mathcal{H}'\Big)\ge 0,
\end{aligned}$$

when k is sufficiently large, since $t_{1,\sigma'}\ge 0$ for $\sigma'\ge 2$. This immediately implies that all terms on the first line are positive.
Step b: Second line. A similar derivation as in Step a gives that

$$\sum_{\sigma'=3}^{s} t_{2,\sigma'}\Big(|R_{\sigma'}|+\frac{1}{2}t_{11}|R_1||R_{\sigma'}|\Big)\ge 0.$$

Concerning $\mathrm{E}\,Y_qY_a$, since $\mathrm{E}\,Y_a=0$, the constant term in Y_q gives no contribution. Furthermore, when we use (56) and

$$\begin{aligned}
\mathrm{E}\,S^2_{R_\sigma}S_{R_0\setminus R_1}S_{R_1} &=0,\quad 1\le\sigma\le s,\\
\mathrm{E}\,S^2_{R_1}S_{R_{\sigma-1}}S_{R_\sigma} &=0,\quad 2\le\sigma\le s,
\end{aligned}$$

we arrive at

$$\begin{aligned}
\mathrm{E}\,Y_qY_a\ \ge\ &\mathrm{E}\Big\{\Big(\sum_{\sigma'=2}^{s}t_{1,\sigma'}S^2_{R_{\sigma'}}\Big)\Big(t_{11}S_{R_0\setminus R_1}S_{R_1}+2\sum_{\sigma=2}^{s}t_{\sigma,\sigma}S_{R_{\sigma-1}}S_{R_\sigma}\Big)\Big\}\\
&+\mathrm{E}\Big\{\Big(t_{1,1}S^2_{R_1}\Big)\Big(\sum_{\sigma=2}^{s}t_{1\sigma}S_{R_\sigma}S_{R_0\setminus R_\sigma}+2\sum_{2\le\sigma<\sigma'\le s}t_{\sigma,\sigma'}S_{R_{\sigma-1}}S_{R_{\sigma'}}\Big)\Big\}=0.
\end{aligned}$$

Step c: Third line. As seen in (59), Y_a consists of a term with negative factors ($t_{\sigma,\sigma}$) and a term with positive factors ($t_{\sigma,\sigma'}$). Writing $(x+y)^3 = x^3 + 3x^2y + 3xy^3 + y^3$ and using (56), we bound $\mathrm{E}\, Y_a^3$ from below by

$$\mathrm{E}\Big(t_{11}S_{R_0\backslash R_1}S_{R_1} + 2\sum_{\sigma=2}^{s} t_{\sigma,\sigma}S_{R_{\sigma-1}}S_{R_\sigma}\Big)^3 \tag{69}$$
$$+\, 3\,\mathrm{E}\Big(t_{11}S_{R_0\backslash R_1}S_{R_1} + 2\sum_{\sigma=2}^{s} t_{\sigma,\sigma}S_{R_{\sigma-1}}S_{R_\sigma}\Big)\Big(\sum_{\sigma=2}^{s} t_{1\sigma}S_{R_\sigma}S_{R_0\backslash R_\sigma} + 2\sum_{2\le\sigma<\sigma'\le s} t_{\sigma,\sigma'}S_{R_{\sigma-1}}S_{R_{\sigma'}}\Big)^2.$$

Since

$$\begin{array}{lcll} \mathrm{E}\, S^3_{R_0\backslash R_1}S^3_{R_1} & = & 0, & \\ \mathrm{E}\, S_{R_{\sigma-1}}S_{R_\sigma}S_{R_{\sigma'-1}}S_{R_{\sigma'}}S_{R_{\sigma''-1}}S_{R_{\sigma''}} & = & 0, & 2\le \sigma,\sigma',\sigma''\le s\ , \\ \mathrm{E}\, S^2_{R_0\backslash R_1}S^2_{R_1}S_{R_{\sigma-1}}S_{R_\sigma} & = & \left\{\begin{array}{ll} 0, & \sigma=2, \\ |R_1||R_{\sigma-1}||R_\sigma|, & 3\le\sigma\le s, \end{array}\right. & \end{array}$$

the first term on the right-hand side of (69) equals

$$12t_{11}t_{22}t_{33}|R_1||R_2||R_3| + 6t_{11}^2\sum_{\sigma=3}^{s} t_{\sigma,\sigma}|R_1||R_{\sigma-1}||R_\sigma|. \tag{70}$$

We use (65) and (66) to obtain that (70) is $\mathcal{O}(\mathcal{H}'^2)$. For the second term of (69), we use Cauchy-Schwarz:

$$\begin{aligned} &\Big|\mathrm{E}\Big(t_{11}S_{R_0\backslash R_1}S_{R_1} + 2\sum_{\sigma=2}^{s} t_{\sigma,\sigma}S_{R_{\sigma-1}}S_{R_\sigma}\Big)\Big(\sum_{\sigma=2}^{s} t_{1\sigma}S_{R_\sigma}S_{R_0\backslash R_\sigma} + 2\sum_{2\le\sigma<\sigma'\le s} t_{\sigma,\sigma'}S_{R_{\sigma-1}}S_{R_{\sigma'}}\Big)^2\Big| \\ \le\ & \Big(\mathrm{E}\Big(t_{11}S_{R_0\backslash R_1}S_{R_1} + 2\sum_{\sigma=2}^{s} t_{\sigma,\sigma}S_{R_{\sigma-1}}S_{R_\sigma}\Big)^2\Big)^{1/2} \\ &\qquad\times\Big(\mathrm{E}\Big(\sum_{\sigma=2}^{s} t_{1\sigma}S_{R_\sigma}S_{R_0\backslash R_\sigma} + 2\sum_{2\le\sigma<\sigma'\le s} t_{\sigma,\sigma'}S_{R_{\sigma-1}}S_{R_{\sigma'}}\Big)^4\Big)^{1/2}. \end{aligned}$$

Using (65), (66) and $(x+y)^2 \le 2(x^2+y^2)$, it is straightforward to show that

$$\begin{aligned} &\Big(\mathrm{E}\Big(t_{11}S_{R_0\backslash R_1}S_{R_1} + 2\sum_{\sigma=2}^{s} t_{\sigma,\sigma}S_{R_{\sigma-1}}S_{R_\sigma}\Big)^2\Big)^{1/2} \\ \le\ & \Big(2\,\mathrm{E}\Big(t_{11}^2S^2_{R_0\backslash R_1}S^2_{R_1} + 4\sum_{\sigma=2}^{s} t^2_{\sigma,\sigma}S^2_{R_{\sigma-1}}S^2_{R_\sigma}\Big)\Big)^{1/2} \\ =\ & 2^{1/2}\Big(t_{11}^2|R_0\backslash R_1||R_1| + 4\sum_{\sigma=2}^{s} t^2_{\sigma,\sigma}|R_{\sigma-1}||R_\sigma|\Big)^{1/2} \le C\mathcal{H}'^{1/2} \end{aligned}$$

and, using $(x+y)^4 \le 8(x^4+y^4)$,

$$\begin{aligned} &\Big(\mathrm{E}\Big(\sum_{\sigma=2}^{s} t_{1\sigma}S_{R_\sigma}S_{R_0\backslash R_\sigma} + 2\sum_{2\le\sigma<\sigma'\le s} t_{\sigma,\sigma'}S_{R_{\sigma-1}}S_{R_{\sigma'}}\Big)^4\Big)^{1/2} \\ \le\ & \Big(8\Big(\sum_{\sigma=2}^{s} t^4_{1\sigma}|R_\sigma|^2|R_0\backslash R_\sigma|^2 + 16\sum_{2\le\sigma<\sigma'\le s} t^4_{\sigma,\sigma'}|R_{\sigma-1}|^2|R_{\sigma'}|^2\Big)\Big)^{1/2} \\ \le\ & C\Big(\sum_{\sigma=2}^{s} t^2_{1\sigma}|R_\sigma||R_0\backslash R_\sigma| + 4\sum_{2\le\sigma<\sigma'\le s} t^2_{\sigma,\sigma'}|R_{\sigma-1}||R_{\sigma'}|\Big), \end{aligned}$$

so that for the third line of (68), we obtain

$$\begin{aligned}
&\frac{1}{8}\sum_{\sigma'=2}^{s}|R_0\backslash R_{\sigma'}||R_{\sigma'}|t_{1,\sigma'}^2+\sum_{2\le\sigma<\sigma'\le s}|R_{\sigma-1}||R_{\sigma'}|t_{\sigma,\sigma'}^2+\mathrm{E}\,Y_a^3/6\\
&\ge(\frac{1}{8}-C\mathcal{H}'^{1/2})\sum_{\sigma'=2}^{s}|R_0\backslash R_{\sigma'}||R_{\sigma'}|t_{1,\sigma'}^2+(1-C\mathcal{H}'^{1/2})\sum_{2\le\sigma<\sigma'\le s}|R_{\sigma-1}||R_{\sigma'}|t_{\sigma,\sigma'}^2-C\mathcal{H}^2\ge 0,
\end{aligned}$$

when k is sufficiently large.
Step d: Fourth line. Concerning $\mathrm{E}\,Y_q(Y_a^2/2+Y_a^3/6+Y_a^4e^{\zeta Y_a}/24)$, clearly

$$\begin{aligned}
&|\,\mathrm{E}\,Y_qY_a^2|\le(\mathrm{E}\,Y_q^4)^{1/4}(\mathrm{E}\,Y_a^6)^{1/3},\\
&|\,\mathrm{E}\,Y_qY_a^3|\le(\mathrm{E}\,Y_q^4)^{1/4}(\mathrm{E}\,Y_a^6)^{1/2}\qquad\text{and}\\
&|\,\mathrm{E}\,Y_qY_a^4e^{\zeta Y_a}|\le(\mathrm{E}\,Y_q^4)^{1/4}(\mathrm{E}\,Y_a^6)^{2/3}(e^{12|Y_a|})^{1/12}.
\end{aligned}$$

Thus, in order to prove that $\mathrm{E}\,Y_q(Y_a^2/2+Y_a^3/6+\mathrm{E}\,Y_qY_a^4e^{\zeta Y_a})=\mathcal{O}(\mathcal{H}'^2)$ for $\underline{t}\in\mathcal{E}$, it is sufficient to prove that $\mathrm{E}\,Y_q^4\le C\mathcal{H}'^4$, $\mathrm{E}\,Y_a^6\le C\mathcal{H}'^3$ and $\mathrm{E}\,e^{12|Y_a|}$ is bounded. We show that, now using (65) and (66),

$$\begin{aligned}
\mathrm{E}\,Y_q^4 &\le C\sum_{\sigma=1}^{s}t_{1,\sigma'}^4|R_{\sigma'}|^4+\sum_{2\le\sigma<\sigma'\le s}t_{\sigma,\sigma'}^4|R_{\sigma'}|^4\\
&\le C\sum_{\sigma=1}^{s}\frac{\mathcal{H}'^2}{|R_{\sigma'-1}|^2|R_{\sigma'}|^2}|R_{\sigma'}|^4+\sum_{2\le\sigma<\sigma'\le s}\frac{\mathcal{H}'^2}{|R_{\sigma-1}|^2|R_{\sigma'}|^2}|R_{\sigma'}|^4\\
&\le C\sum_{\sigma=1}^{s}\mathcal{H}'^2\frac{|R_{\sigma'}|^2}{|R_{\sigma'-1}|^2}+\sum_{2\le\sigma<\sigma'\le s}\mathcal{H}'^2\frac{|R_{\sigma'}|^2}{|R_{\sigma-1}|^2}\le C\mathcal{H}'^2.
\end{aligned}$$

Using similar techniques, we see that $\mathrm{E}\,Y_a^6\le C\mathcal{H}'^3$. Finally, $\mathrm{E}\,e^{12|Y_a|}$ can be bounded, using similar techniques as in the lower bound, together with (65) and (66). We conclude that indeed $|\,\mathrm{E}\,Y_q(Y_a^2/2+Y_a^3/6+Y_a^4e^{\zeta Y_a})|\le C\mathcal{H}'^2$ for all $\underline{t}\in\mathcal{E}$. The four steps together give that $e_2(\underline{t})\ge -C\mathcal{H}^2$, which completes the proof of the lemma.